

THE SYMMETRY OF GRAPHS

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INTRODUCTION

The history of graphs goes back to the work of Euler in his discovery of the equation

$$f - e + v = 2$$

relating the number of faces, edges and vertices of a polyhedron and his treatment of the puzzle of the bridges of Königsberg. Thereafter it grew only slowly, with a paper by Jordan in 1869 and then a number of papers by Cayley. About the beginning of this century it began to attract numbers of workers, frequently to investigations connected more or less closely with the still-unsolved four-colour problem. In 1936 the first book devoted to graph theory appeared (König - Theorie der endlichen und unendlichen Graphen) but there was no book in English until the translation of Berge's book in French was published in 1962. A few other books in English have appeared since.

It was the chance coming of Berge's book into my hands that really determined me to research in the theory of graphs, and my interest in groups and algebra in general that determined that I should make the main subject of this thesis the symmetry of graphs. Chapter 1 is devoted to establishing

a foundation of graph theoretical concepts and results. Apart from the omission, on the grounds of space, of the proofs of some of the simpler results, it has been my aim to make this thesis self-contained as far as the graph-theory is concerned. I have made no such attempt with regard to any algebra or topology I may have used. At the most I have cited references.

Chapter 2 is devoted to automorphism properties of graphs. It includes parts that are well-known and also a considerable proportion of my own researches. These will be indicated in that place.

Chapter 3 recounts briefly the results obtained on the symmetry of graphs in researches that have come to my notice. Those few portions which are my own are again indicated.

The remaining chapters are almost entirely my own. The problem of embedding a graph in a surface, scarcely mentioned here, has been discussed in some few papers by Harary and others, but no-one seems to have considered the problems of symmetric embedding of graphs.

I have to thank most of my colleagues in the Mathematics Department for their help and encouragement, especially my supervisor, Professor G.M. Petersen, who kindly read the whole rough draft of this thesis at a time when he was very busy and suggested many improvements and some problems. I must also mention by name Professor Lawden and Mr. J.C.W. De la Bere, and Professor S.J. Bernau, now of Otago.

CHAPTER I

GRAPHS

1.1 This chapter is devoted to defining and establishing those basic properties of graphs which are required in the later development of the thesis. It contains no new results of any consequence, and hence to keep the chapter as short as possible some proofs are omitted altogether or indicated briefly.

Definition 1.1.1.

A graph G consists of a non-empty set V , whose elements are termed the vertices of G , together with a set E of (non-ordered) pairs of elements of V . The members of E are called the edges of G .

When graphs are defined for different purposes slightly different definitions are sometimes used. In some cases loops (edges defined by a pair of equal elements, that is, one repeated) are allowed. In this thesis the two vertices concerned in an edge are always distinct. Nor may more than one edge join any two given vertices. The edge defined by the two vertices a and b is denoted by (a,b) or (b,a) , since these mean the same thing. The edge is said to join the vertices and to be incident with each of them. Two vertices joined by

an edge are said to be adjacent.

The vertices of a graph are represented diagrammatically as points and the edges as (not necessarily straight) line segments.

Definition 1.1.2.

The order of a graph is the number of its vertices.

A graph of finite order is referred to simply as a finite graph, and a graph of infinite order as an infinite graph.

Definition 1.1.3.

The valency of a vertex a of a graph G is the number of edges incident with a . (The valency is called the degree by many authors.)

Theorem 1.1.1.

Let G be a finite graph, and for each non-negative integer r let v_r be the number of vertices of G of valency r . Then if v is the total number of vertices in G , and e the total number of edges,

$$\begin{aligned} v &= v_0 + v_1 + v_2 + \dots + v_r + \dots \\ 2e &= v_1 + 2v_2 + \dots + rv_r + \dots \end{aligned}$$

Both sums are in fact finite, since the number of vertices adjacent to any vertex is at most $v - 1$. Thus for $r \geq v$, v_r is necessarily zero.

1.2 Two families of graphs require special mention. Although the set of vertices is by definition not empty, the set of edges may be empty, giving rise to the family of null graphs N_n , with n vertices and no edges for each positive integer n .

On the other hand if each possible pair of vertices does define an edge we obtain a complete graph K_n on n vertices for each positive integer n . The valency of each vertex is then $n-1$, and the number of edges is $\frac{1}{2}n(n-1)$. Conversely any graph with n vertices and $\frac{1}{2}n(n-1)$ edges is complete.

We notice that in the special case when $n = 1$, both N_1 and K_1 consist of a single isolated vertex.

1.3 Definition 1.3.1.

Let G be any graph, V its set of vertices and E its set of edges and V' a subset of V . Then the graph G' whose vertices form the set V' and whose set of edges E' consists of those members of E both of whose endpoints (vertices) are members of V' , is termed a subgraph of G .

Each graph is clearly a subgraph of itself; a subgraph which is not the whole graph will be termed proper.

Definition 1.3.2.

Let G be any graph, V its set of vertices and E its

set of edges. Then a graph G' which has V as its set of vertices and E' its set of edges a subset of E , is called a partial graph of G .

These two ideas may be combined in defining a partial subgraph.

Definition 1.3.3.

Let G be any graph, V its set of vertices, E its set of edges. Then a graph G' with set V' of vertices a subset of V , and E' its set of edges a subset of E , with the additional property that every edge in E' joins two vertices in V' , is called a partial subgraph of G .

A partial subgraph of G may be regarded either as a partial graph of a subgraph of G or as a subgraph of a partial graph of G .

1.4 Definition 1.4.1.

Let a_1, a_2, \dots, a_n be a sequence of n vertices (not necessarily distinct, but $a_r \neq a_{r+1}$ for $r = 1, 2, \dots, n-1$) in a graph G such that each pair (a_r, a_{r+1}) defines an edge of G . Then the sequence

$$(a_1, a_2), (a_2, a_3), \dots, (a_{n-1}, a_n)$$

of $n-1$ edges is called a path in G , from a_1 to a_n .

A path in which $a_r \neq a_s$ if $r \neq s$ is termed elementary.

The case where $n = 1$ may be considered to give rise to a trivial elementary path from a_1 to a_1 consisting of no edges. The number of edges in a path is termed its length.

Theorem 1.4.1.

Every path between two vertices a and b contains an elementary path between them.

If a and b coincide there is the trivial elementary path. Otherwise construct from the edges of the path the list of vertices in order, as in the definition. If no vertex is repeated, the path is already elementary. If $a_r = a_s$ for some r, s with $1 \leq r < s \leq n$, consider the path

$$(a_1, a_2)(a_2, a_3) \dots (a_{r-1}, a_r), (a_s, a_{s+1}), \dots, (a_{n-1}, a_n),$$

which has list of vertices

$$a_1, a_2, \dots, a_r = a_s, \dots, a_n.$$

This is necessarily shorter than the original path. The formal proof is then completed by induction on n .

Theorem 1.4.2.

Let P be the relation on the vertices of a graph G defined by

aPb if and only if there is a path in G from a to b .

Then P is an equivalence relation.

aPa at once from the definition of trivial paths of length zero.

If $a P b$, there is a path $(a, a_1), (a_1, a_2), \dots, (a_n, b)$ in G . There is therefore also a path $(b, a_n), (a_n, a_{n-1}), \dots, (a_2, a_1), (a_1, a)$ since these are the same edges, and we again get a sequence of vertices with neighbouring members of the sequence adjacent in the graph. Thus $b P a$, and P is symmetric.

Finally, to prove P transitive we juxtapose paths. Given the paths $(a, a_1), (a_1, a_2), \dots, (a_n, b)$ from a to b and $(b, b_1), (b_1, b_2), \dots, (b_m, c)$ from b to c we join them to produce the path $(a, a_1), (a_1, a_2), \dots, (a_n, b), (b, b_1), \dots, (b_m, c)$ from a to c .

Since P is an equivalence relation it divides the set of vertices V of G into disjoint equivalence classes.

Definition 1.4.2.

Let V' be an equivalence class defined in the graph G by the above relation P . Then the subgraph G' defined by V' is called a component of G .

Definition 1.4.3.

A graph G such that given any two vertices a, b in G there is a path from a to b is called connected.

This definition coincides with the topological definition if the edges are thought of as arcs, homeomorphic images of the unit line segment.

A connected graph has only one component.

Theorem 1.4.3.

Each component of a graph is connected.

This follows at once from the definition of the components as equivalence classes. Thus a and b belong to the same component only if there is a path from a to b .

Theorem 1.4.4.

Every graph with a connected partial graph is connected.

For each pair of vertices in the given graph is a pair in the partial graph, and the path between them in the partial graph is also a path in the given graph.

Theorem 1.4.5.

If a and b are vertices belonging to different components of a graph G , there is no path from a to b .

For if there is a path from a to b , a and b belong to the same equivalence class relative to P , and so to the same component.

Theorem 1.4.6.

If a and b are vertices belonging to different components of a graph G , a is not adjacent to b .

For if a is adjacent to b , (a,b) is an edge of G and so a path of length 1 from a to b .

Definition 1.4.4.

If a is a vertex of the graph G and a has the valency

0, the subgraph consisting of a alone and no edges is a component of G . The vertex a is then said to be isolated.

The null graph N_n consists of n isolated vertices. On the other hand, since every pair of vertices defines an edge in the complete graph K_n , K_n is connected for each n .

1.5 If for each pair of vertices a, b in a connected graph G we define $d(a, b)$ as the minimum length of all paths from a to b we can establish very simply that $d(a, b)$ is a metric on the vertices of G . We can show also that we need only take the minimum over elementary paths.

Definition 1.5.1.

If G is a connected graph, the maximum of $d(a, b)$ over all pairs of vertices a, b in G is called the diameter of G .

It may be readily shown that the diameter is 0 if and only if G is an isolated vertex, and 1 if and only if G is a complete graph (possibly of infinite order).

1.6 Definition 1.6.1.

Let G and H be two graphs, and let ϕ be a one-to-one mapping of the vertices of G onto the vertices of H . Then if ϕ has the additional property that $(\phi(a), \phi(b))$ is an edge in H if and only if (a, b) is an edge in G for all pairs of vertices a, b in G , we say that ϕ is an isomorphism of G onto H , and that G is isomorphic to H .

The following theorems may be easily proved.

Theorem 1.6.1.

If G and H are two graphs with G isomorphic to H , then H is isomorphic to G .

Theorem 1.6.2.

If G, H, K are three graphs with G isomorphic to H and H isomorphic to K , G is isomorphic to K .

Theorem 1.6.3.

Every graph is isomorphic to itself.

Theorem 1.6.4.

If G and H are isomorphic graphs they have the same order.

Theorem 1.6.5.

If G and H are isomorphic they have the same number of edges.

Theorem 1.6.6.

Two null graphs are isomorphic if and only if they have the same order.

Theorem 1.6.7.

Two complete graphs are isomorphic if and only if they have the same order.

These two theorems are important since they allow us to talk about 'the null graph N_n ' and 'the complete graph K_n ' when the properties under discussion are isomorphism invariants, as will normally be the case.

Theorem 1.6.8.

If $(a_1, a_2), (a_2, a_3), \dots, (a_{n-1}, a_n)$ is a path in a graph G , and ϕ is an isomorphism of G onto a graph H ,
 $(\phi(a_1), \phi(a_2)), (\phi(a_2), \phi(a_3)), \dots, (\phi(a_{n-1}), \phi(a_n))$ is a path in H .

This enables us to prove

Theorem 1.6.9.

The components of the graph G are mapped onto the components of H by any isomorphism from G onto H .

From the definition of components (Defn 1.4.2) this is equivalent to showing that there is a path from $\phi(a)$ to $\phi(b)$ in H if and only if there is a path from a to b in G . To the previous theorem we need only add that if ϕ is an isomorphism so is the inverse mapping ϕ^{-1} of H onto G .

Theorem 1.6.10.

If G is connected and G is isomorphic to H , H is also connected.

This is a simple corollary of Theorem 1.6.9.

A number of necessary conditions may easily be obtained, in addition to the above, for two graphs to be isomorphic.

They must for instance have equal numbers of vertices of each valency, and also equal numbers of edges joining a vertex of valency r to a vertex of valency s for each r and each s . But this condition, together with connectedness is not a sufficient condition for isomorphism. Several other criteria will be considered later, but it does not seem that any combination of these will form a general necessary and sufficient criterion for isomorphism.

In general in this thesis we will not distinguish between isomorphic graphs.

1.7 Definition 1.7.1.

Let G be a finite graph with vertices a_1, a_2, \dots, a_n . We define a second graph \bar{G} with vertices b_1, b_2, \dots, b_n and edges determined by the rule

(b_r, b_s) is an edge of \bar{G} if and only if (a_r, a_s) is not an edge of G .

We call \bar{G} the complement of G .

Theorem 1.7.1.

The complement of the complement of G is isomorphic to G .

Theorem 1.7.2.

If G and H are isomorphic finite graphs, their complements \bar{G} and \bar{H} are isomorphic.

The proofs of both these theorems are immediate. We can also show from the definition of each that

Theorem 1.7.3.

The complement of a complete graph of finite order is null.

The complement of a null graph of finite order is complete.

More interesting is the next theorem, which is not immediately obvious (in the non-mathematical sense).

Theorem 1.7.4.

Let G be a non-connected finite graph. Then its complement \bar{G} is connected, and has diameter at most 2.

Since G is not connected, it must contain at least two components. If the vertices a_i, a_j of G belong to different components, (a_i, a_j) is not an edge of G (Th.1.4.6) so (b_i, b_j) is an edge of \bar{G} .

If on the other hand a_i and a_j belong to the same component, there is a third vertex a_k which does not belong to this component. The edges (b_i, b_k) and (b_j, b_k) thus belong to \bar{G} , and form a path linking b_i and b_j . Thus in either case there is a path from b_i to b_j , of length either 1 or 2.

It is of course possible for a graph and its complement both to be connected. In this case both may have diameters greater than 2. In the self-complementary graphs discussed in the next section the diameter is always either 2 or 3. We can complete the discussion of diameters of

complementary pairs of graphs by proving the theorem:

Theorem 1.7.5.

Let G be a connected finite graph of diameter $d \geq 4$. Then \bar{G} , the complement of G , is connected and has diameter precisely 2.

Let G have vertices a_1, a_2, \dots, a_n and let \bar{G} have corresponding vertices b_1, b_2, \dots, b_n .

Then if G has diameter d there is a path $(a_1, a_2), (a_2, a_3), \dots, (a_d, a_{d+1})$ of length d , which is a shortest path in G between a_1 and a_{d+1} for a suitable numbering of the vertices of G and \bar{G} . We consider the vertices a_1, a_2, a_3, a_4, a_5 . The edges $(a_1, a_2), (a_2, a_3), (a_3, a_4), (a_4, a_5)$ belong to the above path and so to G . But as a_5 is a vertex in a shortest path between a_1 and a_{d+1} , there is no path from a_1 to a_5 with less than four edges. Hence none of the other edges between these five vertices belong to G . So the corresponding pairs $(b_1, b_3), (b_1, b_4), (b_1, b_5), (b_2, b_4), (b_2, b_5), (b_3, b_5)$ do determine edges in \bar{G} . Considering each pair of b_1, b_2, b_3, b_4, b_5 in turn, we can find a path of length at most 2 between each pair.

If G has order 5 there are no more vertices. If G has more vertices, let any one of these be a_r . As the shortest path from a_1 to a_5 has length 4, not both (a_1, a_r) and (a_5, a_r) are edges in G , so at least one of (b_1, b_r) and (b_5, b_r) is an edge in \bar{G} , and we can find a path of length at most 2 from b_r to any of b_1 to b_5 .

If G has order 6 there are no more vertices. Otherwise let a_s be any other vertex in G . As in the previous paragraph there is a path of length at most 2 from b_s to each of b_1, b_2, b_3, b_4, b_5 . It remains only to find a path of length at most 2 between b_r and b_s .

We consider two cases: If (a_r, a_s) is not an edge of G , (b_r, b_s) is an edge of \bar{G} , so there is a path between them of length 1. On the other hand if (a_r, a_s) is an edge of G , we consider the four pairs $(a_1, a_r), (a_1, a_s), (a_5, a_r), (a_5, a_s)$ and the corresponding pairs in \bar{G} . Considering each of the possible cases we find that unless at least one of a_1, a_5 is not adjacent to either a_r or a_s , there is a path of length 2 or 3 from a_1 to a_5 in G . Hence we have that at least one of b_1, b_5 is adjacent to both b_r and b_s , yielding a path of length 2 between them.

To complete the proof we have to show that the diameter is at least 2. But as we remarked in section 1.5, only the complete graphs have diameter 1, and hence if \bar{G} has diameter 1 its complement is null: but G is not null, being connected and of order at least 5.

1.8 Definition 1.8.1.

Let G be a finite graph. Then if G is isomorphic to its complement \bar{G} , G is said to be self-complementary.

Self-complementary graphs have been studied in papers

by Ringel (Ri5) and Sachs (Sa5), with considerable overlap between them, and by Read (Re1). We outline a few of the basic results here:

Theorem 1.8.1.

A self-complementary graph of order n has $\frac{1}{2}n(n-1)$ edges.

Because the graph and its complement must have the same number of edges, and the total must be $\frac{1}{2}n(n-1)$, the number in the complete graph of order n .

Theorem 1.8.2.

Every self-complementary graph has order n congruent to either 1 or 0 modulo 4.

This is necessary to make $\frac{1}{2}n(n-1)$ integral.

Theorem 1.8.3.

Every self-complementary graph is connected.

This is a direct consequence of Theorems 1.6.10 and 1.7.4.

Theorem 1.8.4.

Every self-complementary graph has diameter 2 or 3.

This is a direct corollary of the previous theorem and theorem 1.7.5, since isomorphic graphs have the same diameter.

This result was proved directly by Ringel.

Self-complementary graphs of each of the orders

satisfying the necessary condition of theorem 1.8.2 do exist, and some are of diameter 2 and some of diameter 3, at least one of each for all orders greater than or equal to 5. (Ringel).

Ringel and Sachs both investigated the isomorphism mapping G onto \bar{G} . Considering this as a permutation of the suffices they found that it consisted of disjoint cycles of lengths a multiple of four, together with a fixed suffix if the order is odd. Conversely if P is any such permutation it corresponds to at least one self-complementary graph in this fashion.

Read's paper gives the number of non-isomorphic self-complementary graphs for each order. His table for the first few orders reads

4	5	8	9	12	13	16	17
1	2	10	36	720	5600	703760	11220000

The graphs for orders up to 8 are shown in fig. 1.8.1.

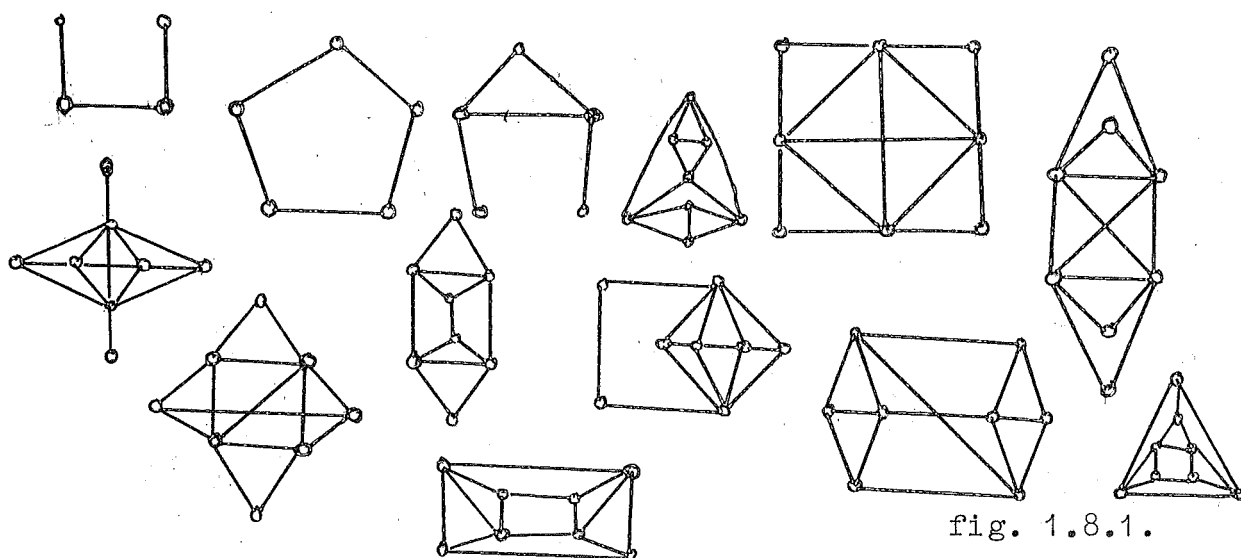


fig. 1.8.1.

1.9 Referring back to section 1.4, we consider those paths both of whose endpoints are the same.

Definition 1.9.1.

Let a_1, a_2, \dots, a_n be a sequence of n vertices (not necessarily distinct but with $a_r \neq a_{r+1}$ for $r = 1, 2, \dots, n$) in a graph G , such that each pair (a_r, a_{r+1}) defines an edge in G , and (a_n, a_1) is also an edge in G .

Then the sequence

$$\{(a_1, a_2), (a_2, a_3), \dots, (a_{n-1}, a_n), (a_n, a_1)\}$$

of n edges in G is called a circuit.

A circuit in which no edge appears more than once is termed elementary. A circuit need not contain an elementary circuit. The length of a circuit is the number of edges it contains.

Definition 1.9.2.

A connected finite graph without any elementary circuits is called a tree.

The orem 1.9.1.

Let H be a graph of order $n > 1$. Then any one of the following equivalent properties characterises a tree:

- (i) H is connected and does not possess any elementary circuits;
- (ii) H has no elementary circuits and has $(n-1)$ edges;

- (iii) H is connected and has $n-1$ edges;
- (iv) H contains no elementary circuits, and if an edge is added which joins two non-adjacent vertices, one and only one elementary cycle is formed;
- (v) H is connected but loses this property if any edge is deleted;
- (vi) every pair of vertices in H is joined by one and only one elementary path.

This is precisely - but for changes of notation - theorem 1 of Chapter 16 of Berge (Be1).

Definition 1.9.3.

By a continuation of the botanical analogy any graph all of whose components are trees is called a forest.

Theorem 1.9.2.

Let G be a tree with at least two vertices. Then G has at least two vertices with valency 1.

For consider the two equations of theorem 1.1.1. Since we have here n vertices and $n-1$ edges these equations become

$$n = v_0 + v_1 + v_2 + v_3 + \dots \quad (i)$$

$$2(n-1) = v_1 + 2v_2 + 3v_3 + \dots \quad (ii)$$

Since G is connected and $n \geq 2$, $v_0 = 0$. Then subtracting (ii) from twice (i),

$$2 = v_1 - v_3 - 2v_4 - 3v_5 - \dots \quad (iii)$$

and this equation requires that v_1 be at least 2, since no v_r is negative.

This proof may be compared with the rather less satisfactory proof in Berge (Theorem 2 of chapter 16).

Definition 1.9.4.

A vertex in any graph with valency 1 is called a pendant or terminal vertex, and the edge incident with it a pendant edge.

Definition 1.9.5.

Let T be any tree, N the set of non-pendant vertices in T . Then the subgraph U of T determined by N is called ' T -pruned', provided that N is not empty. If N is empty, or T consists of an isolated vertex, T -pruned is not defined.

Theorem 1.9.3.

T -pruned is a tree.

Let p be the number of pendant vertices in T , n the total number. Then U has $n-p$ vertices. Also U has all those edges of T which are not incident with pendant vertices in T , that is, all the non-pendant edges of T . But there is a one-to-one correspondence between the pendant edges and pendant vertices except in the single case when T has two vertices and one edge, and in this case T -pruned is not defined. Hence there are also p pendant edges, and $n-p-1$ edges in U .

If a and b are two vertices of U , there is an elementary path between them in T as T is connected. But no pendant vertex, and hence no pendant edge can appear in an elementary path, so this path lies wholly in U . Hence U is connected, and using criterion (iii) of theorem 1.9.1 is a tree.

The process of pruning a tree may be continued to provide a sequence of trees T_0, T_1, T_2, \dots where each T_r is T_{r-1} -pruned. As pruning removes at least two vertices at each stage, the order is reduced by two or more each time and hence finally one or other of the non-prunable trees is obtained, the trees (unique within isomorphism) with one or two vertices.

If the final pruned tree is an isolated vertex, this vertex is termed the centre of all the trees in the sequence, if the final tree has two vertices they are termed the bicentres of each tree in the sequence.

There are several other properties which lead to a centre or pair of bicentres for a tree. One is to associate with each vertex the length of the longest elementary path beginning at that vertex. Another is the property of moments to be described in the next section. The centres and bicentres defined by different methods do not necessarily coincide, and a graph may have a centre according to one definition and a pair of bicentres according to the other.

1.10 Definition 1.10.1.

Let G be a finite connected graph, with vertices a_1, \dots, a_n . Define d_{ij} to be the distance between a_i and a_j (length of the shortest path between them). Then $m_i = \sum_{j=1}^n d_{ij}$ is termed the moment of the graph G about the vertex a_i .

Moments were first defined in my paper (Ro4) in a more general sense, where there is a non-negative real weight associated with each vertex, and a positive real length associated with each edge. In the more restricted definition used here the weight of each vertex is taken as 1, and the length of each edge also as 1. A vertex is a minimum if the moment of G about the vertex is less than or equal to the moments about all adjacent vertices. Even in the general case the minima are all together (i.e. any path between minima passes only through minima) for any tree, and if the above restricted definition is used there are at most two minima in any tree. The minimum is therefore another centre-bicentre property.

1.11 In one or two places we shall require the idea of a homeomorphism of one graph onto another. We first define a topology on G by taking the vertices with the discrete topology and the edges as homeomorphic images of the unit interval.

It is clear that if p is any point on an edge but not a vertex, then p has a small neighbourhood homeomorphic to

an open interval, and whose boundary consists of two isolated points. If on the other hand q is a vertex of valency r , a small neighbourhood of q has a boundary consisting of r points. Thus for $r \neq 2$ every homeomorphism of the graph maps each vertex of valency r onto a vertex of valency r . But vertices of valency 2 have no such permanence. For instance all graphs with $n(\geq 3)$ vertices a_1, \dots, a_n and edges (a_i, a_{i+1}) for $i = 1, \dots, n-1$ and (a_1, a_n) are homeomorphic to each other.

CHAPTER II

AUTOMORPHISMS

2.1 We remarked in theorem 1.6.3 that every graph was isomorphic to itself, and we prove this by reference to an identity mapping sending each vertex onto itself. But this may not be the only way to map the graph onto itself by an isomorphism.

Definition 2.1.1.

An automorphism of a graph G is an isomorphism of G onto itself, that is, a permutation of the vertices of G such that $(\phi(a), \phi(b))$ is an edge if and only if (a, b) is an edge.

For every finite graph every isomorphism into itself is necessarily onto, since an isomorphism is one-to-one. But it is possible for an infinite graph to be isomorphic to a proper subgraph of itself. Such a mapping is not considered an automorphism. An example is the graph with the vertices $a_1, a_2, \dots, a_n, \dots$, and edges (a_n, a_{n+1}) for all positive integers n , with isomorphism into itself $\phi(a_n) = a_{n+1}$ for all n .

Every graph must have the identity automorphism ϵ ; $\epsilon(a) = a$ for each vertex a .

(Automorphisms will always be denoted by Greek letters). A graph which has no other automorphisms will be termed asymmetric. Apart from the trivial example of a graph consisting of a single vertex, the simplest asymmetric graphs have at least six vertices: there are then several examples. Erdős and Renyi have written a paper on asymmetric graphs (ER2).

2.2 Given two automorphisms ϕ, ψ of a graph G we define the product $\phi\psi$ by $\phi\psi(a) = \phi(\psi(a))$ for each vertex a of G . As ϕ and ψ are permutations on the set of vertices, and the collection of all these permutations is a group with this operation, $\phi\psi$ is also a permutation on the vertices of G . It is also an automorphism of G since

- (i) $(\phi(\psi(a)), \phi(\psi(b)))$ is an edge if and only if $(\psi(a), \psi(b))$ is an edge, and
- (ii) $(\psi(a), \psi(b))$ is an edge if and only if (a, b) is an edge.

The inverse ϕ^{-1} of ϕ as an automorphism is the inverse of ϕ as a permutation, defined by taking $\phi^{-1}(a)$ as b , where $a = \phi(b)$. And the definition of an automorphism, $(\phi(a), \phi(b))$ is an edge if and only if (a, b) is an edge can be reworded, taking $\phi(a) = c$, $\phi(b) = d$ as (c, d) is an edge if and only if $(\phi^{-1}(c), \phi^{-1}(d))$ is an edge. Hence ϕ^{-1} is also an automorphism of G .

The collection Γ of automorphisms of G is therefore a subgroup of the collection of permutations on the vertices of G and hence:

Theorem 2.2.1.

The collection of automorphisms of G under the operation of composition is a group.

This group is frequently referred to simply as the group of the graph.

Since the automorphism group is a subgroup of the symmetric group of all permutations of the vertices, its order divides the order of the symmetric group. Hence:

Theorem 2.2.2.

The order of the automorphism group of a finite graph of order n divides $n!$

Theorem 2.2.3.

Let G be a graph of finite order n , and let G have as automorphism group the symmetric group on its n vertices. Then G is either the complete graph K_n or the null graph N_n .

For let a_1, a_2, b_1, b_2 , be vertices, a_1 distinct from a_2 , b_1 distinct from b_2 , but not otherwise necessarily different. Then if G has the symmetric group it has an automorphism which maps a_1 to b_1 and a_2 to b_2 . Then if (a_1, a_2) is an edge, so is (b_1, b_2) . Hence if any pair defines an edge, every pair defines an edge. So either G has no edges and is null or all pairs are edges and G is complete.

Conversely consider N_n . Let ϕ be any permutation on the vertices and a_1, a_2, b_1, b_2 be as above then neither

(a_1, a_2) nor (b_1, b_2) is an edge, and ϕ satisfies the automorphism conditions. Similarly in K_n both (a_1, a_2) and (b_1, b_2) are edges and ϕ again satisfies the automorphism condition.

Theorem 2.2.4.

Let G and H be two isomorphic graphs. Then G and H have isomorphic automorphism groups.

Let ϕ be the isomorphism from G onto H . Then the inverse mapping ϕ^{-1} is an isomorphism from H onto G . Now let ψ be an automorphism of G . Let a, b be vertices in H , and let (a, b) be an edge of H . Then as ϕ^{-1} is an isomorphism, $(\phi^{-1}(a), \phi^{-1}(b))$ is an edge of G . As ψ is an automorphism of G , $(\psi\phi^{-1}(a), \psi\phi^{-1}(b))$ is also an edge of G , and finally since ϕ is an isomorphism, $(\phi\psi\phi^{-1}(a), \phi\psi\phi^{-1}(b))$ is an edge of H . Similarly if (a, b) is not an edge of H , neither is $(\phi\psi\phi^{-1}(a), \phi\psi\phi^{-1}(b))$. Hence $\phi\psi\phi^{-1}$ is an automorphism of H . If ψ_1 and ψ_2 are two automorphisms of G , $\phi\psi_1\phi^{-1}$ and $\phi\psi_2\phi^{-1}$ are automorphisms of H and $(\phi\psi_1\phi^{-1})(\phi\psi_2\phi^{-1})$ equals $\phi\psi_1\psi_2\phi^{-1}$, showing that H has a group of automorphisms isomorphic to the group of G . That is, the group of G is isomorphic to some subgroup of the group of H . But we may reverse the roles of G and H to show in the same fashion that the group of H is isomorphic to some subgroup of the group G . Hence the groups of G and H are isomorphic.

On the other hand it is easy to find pairs of graphs

with isomorphic groups which are not themselves isomorphic. Indeed as we show in section 2.4 a graph and its complement have equal permutation groups of automorphisms (that is, the same permutations are automorphisms of both), yet they are not in general isomorphic.

2.3 The above discussion applies equally to finite and infinite graphs, but with infinite graphs there is also a possibility of isomorphisms strictly into itself. The collection of all isomorphisms into, including those which are onto, forms a semigroup. The members of the semigroup which have inverses are precisely the automorphisms of the graph. The automorphism group may be either finite or infinite.

But every strictly-into isomorphism is of infinite order. Let ϕ be such an isomorphism, and consider $\phi^2, \phi^3, \dots, \phi^n, \dots$. As ϕ is strictly into there exists a vertex b such that there is no vertex a with $\phi(a) = b$. If for some n , $\phi^n = \epsilon$, the identity, $\phi\phi^{n-1} = \epsilon$, so $\phi(\phi^{n-1}(b)) = \epsilon(b) = b$, yielding a contradiction.

2.4 Theorem 2.4.1.

Let G be a finite graph with vertices a_1, \dots, a_n and \bar{G} its complement with corresponding vertices b_1, \dots, b_n . Then G and \bar{G} have equal automorphism groups, considered as permutations on the suffices $1, \dots, n$. For the automorphism condition on G is that $(\phi(a_r), \phi(a_s))$ is an edge in G if and

only if (a_r, a_s) is an edge in G . This is the same as saying that $(\phi(a_r), \phi(a_s))$ is not an edge in G if and only if (a_r, a_s) is not an edge in G . And this is the same as saying that $(\phi(b_r), \phi(b_s))$ is an edge in \bar{G} if and only if (b_r, b_s) is an edge in \bar{G} .

2.5 Suppose a graph G consists of a number of components G_1, G_2, \dots, G_n . Each of these may be considered as a graph in its own right and will have automorphisms. In this section we consider the relationships between the group Γ of the whole graph and the groups $\Gamma_1, \dots, \Gamma_n$ of the components.

Theorem 2.5.1.

Let ϕ be an automorphism of G , and a, b two vertices of G . Then there is a path from $\phi(a)$ to $\phi(b)$ if and only if there is a path from a to b .

If there is a path from a to b there exist vertices a_1, \dots, a_n such that $(a, a_1), (a_1, a_2), \dots, (a_n, b)$ are edges in G . Then under ϕ these edges are mapped to $(\phi(a), \phi(a_1)), (\phi(a_1), \phi(a_2)), \dots, (\phi(a_n), \phi(b))$, forming a path from $\phi(a)$ to $\phi(b)$. Indeed this path is of the same length as the path from a to b , and is elementary if and only if the original path is elementary.

The converse part of the result is established by considering the inverse automorphism ϕ^{-1} in the same way.

We can then establish at once as a corollary:

Theorem 2.5.2.

Let ϕ be an automorphism of G , and a, b two vertices of G . Then $\phi(a)$ and $\phi(b)$ belong to the same component of G if and only if a and b belong to the same component of G .

This is merely a translation of the previous theorem employing the definition of a component.

The automorphisms of G therefore preserve components, and each generates a permutation of the components.

Theorem 2.5.3.

If G_1 and G_2 are components of G , there is an automorphism of G which maps G_1 onto G_2 if and only if G_1 and G_2 are isomorphic.

Suppose there is an automorphism ϕ of G which maps component G_1 onto component G_2 . Then for every pair a, b in G_1 there is a pair $\phi(a), \phi(b)$ in G_2 , and either both define edges or neither does. That is, ϕ restricted to G_1 is an isomorphism of G_1 onto G_2 . Thus there is an automorphism of G onto G_2 only if G_1 and G_2 are isomorphic.

Conversely, let G_1 and G_2 be isomorphic components of G . Then there is an isomorphism ψ mapping G_1 onto G_2 and its inverse ψ^{-1} maps G_2 onto G_1 . We now define an automorphism ϕ of G by

$$\phi(a) = \psi(a) \quad \text{if } a \in G_1$$

$$\phi(a) = \psi^{-1}(a) \quad \text{if } a \in G_2$$

$$\phi(a) = a \quad \text{otherwise.}$$

That this is an automorphism is shown at once by considering the nine cases of vertices a and b belonging to G_1 or G_2 or some other component.

We consider next the case of a graph G with a finite collection of components G_1, \dots, G_m , no two being isomorphic. Then in virtue of the above theorem we know that each automorphism of G maps each component onto itself. If then we denote by ϕ_r the automorphism ϕ restricted to the component G_r , ϕ_r is an automorphism of G_r .

Theorem 2.5.4.

Let G be a graph with components G_1, G_2, \dots, G_m , no two isomorphic, and let G_r have automorphism group Γ_r for each $r = 1, \dots, m$. Then the automorphism group Γ of G is the direct product

$$\Gamma = \Gamma_1 \otimes \Gamma_2 \otimes \dots \otimes \Gamma_m.$$

Let ψ_r be an automorphism of G_r , and define ψ'_r over G by

$$\psi'_r(a) = \psi_r(a) \text{ if } a \in G_r$$

$$\psi'_r(a) = a \text{ if } a \notin G_r.$$

Then ψ'_r is an automorphism of G . Moreover if $s \neq r$ $\psi'_r \psi'_s = \psi'_s \psi'_r$, for any ψ'_r in Γ_r and ψ'_s in Γ_s . Also any automorphism ψ of G can be written uniquely as a product $\psi'_1 \psi'_2 \dots \psi'_m$, where ψ'_r for each r corresponds to ψ_r , which is in turn ψ restricted to G_r . The result is therefore established.

To go to the other extreme, suppose that G_1, G_2, \dots, G_m the components of G are all isomorphic. Then they all have isomorphic automorphism groups. There is also at least one automorphism of G generating any permutation π on the components. This group is called the wreath product $S_m(\Gamma')$, where Γ' is the automorphism group of any simple component of G and S_m is the symmetric group on m objects. In general we have the following definition for finite groups:

Definition 2.5.1.

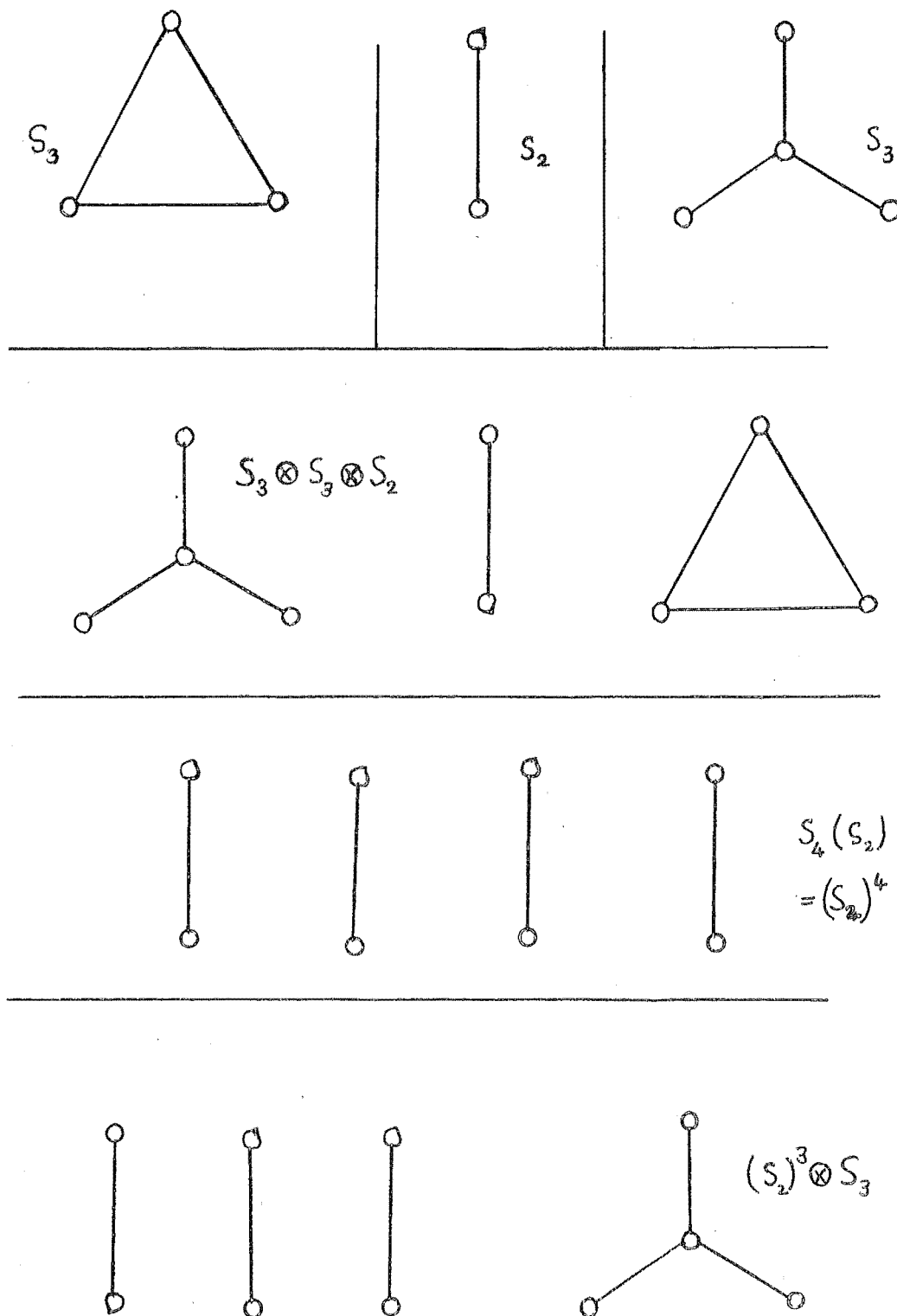
Let Γ and K be permutation groups with Γ of order g and degree r and K of order k and degree s . We first define s isomorphic copies $\Gamma_1, \dots, \Gamma_s$ of Γ and then take a set of rs objects x_{ij} with $i = 1, \dots, s$ and $j = 1, \dots, r$. Then we consider the collection of all permutations of the type $\phi(\kappa, \gamma_1, \gamma_2, \dots, \gamma_s)$ which maps x_{ij} to x_{pq} where $p = \kappa(i)$ and $q = \gamma_i(j)$ and κ is a member of K and each γ_i is a member of Γ_i . We define this group to be wreath product $K(\Gamma)$.

The above wreath product $K(\Gamma)$ has order kg^s , so that when K is S_n and Γ is the group, of order g , of a component of a graph with n isomorphic components, the group $S_n(\Gamma)$ has order $n!g^n$.

This product of two graphs was introduced by Polya in (Po2 p 178). See also the paper (Fr 5) by Frucht.

To shorten the notation I have adopted the notation Γ^n to be the same as $S_n(\Gamma)$.

fig. 2.5.1.



In the more general case than the two extremes treated above, the group of the whole graph is a direct product of wreath products of the groups of the components. The possibilities are illustrated in fig. 2.5.1.

Wreath products of other kinds arise also in a direct fashion. Figures 2.5.2 illustrates this with the wreath product of the dihedral group D_4 and the cyclic group C_2 . Here four subgraphs each with group C_2 can only be permuted according to the dihedral group on four objects.

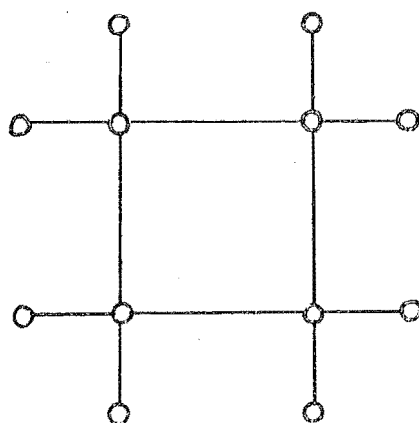


fig. 2.5.2.

2.6 Theorem 2.6.1.

Let G be any graph with automorphism group Γ and let $\Phi(a)$ be the set of all automorphisms of G which hold the vertex a fixed, that is

$$\phi \in \Phi(a) \text{ if and only if } \phi(a) = a.$$

Then $\Phi(a)$ is a subgroup of Γ .

For if ϕ_1, ϕ_2 belong to $\Phi(a)$

$$\phi_1\phi_2(a) = \phi_1(\phi_2(a)) = \phi_1(a) = a$$

and $\phi_1(a) = a$ implies $\phi^{-1}(a) = a$.

Also if ϵ is the identity $\epsilon(a) = a$, so $\Phi(a)$ is not empty.

This subgroup may consist of the identity only or it may in the other extreme be the whole group Γ . $\Phi(a)$ is called the fixing subgroup of a . A vertex for which the fixing subgroup is the whole automorphism group is called a fixed vertex. A graph without fixed vertices is called fixed point free.

The definition may be extended to any set of vertices in G .

Definition 2.6.1.

Let S be a set of vertices in G . Then $\Phi(S)$ is the set of automorphisms defined by

$$\phi \in \Phi(S) \text{ if and only if } \phi(a) = a \text{ for all } a \in S.$$

Theorem 2.6.2.

$$\Phi(S) = \bigcap_a \Phi(a) \text{ over all } a \in S.$$

The proof is immediate and implies that $\Phi(S)$ is also a subgroup of Γ .

To turn to the converse property, let Ψ be any subgroup of Γ . Then we define $F(\Psi)$ to be the set of vertices

held fixed by every member of Ψ . Formally:

Definition 2.6.2.

If Ψ is a subgroup of Γ , we define $F(\Psi)$ by

$a \in F(\Psi)$ if and only if $\psi(a) = a$ for all $\psi \in \Psi$.

It may be, of course, that F is empty.

Now suppose we take any set S of vertices in G . This will determine a fixing subgroup $\Phi(S)$, and this subgroup in turn determines a set of vertices $F(\Phi(S))$. Clearly this set contains S , but it may contain other vertices besides. Thus in fig. 2.6.1, a_1 is held fixed by every automorphism of the graph - it is a fixed vertex. Every automorphism which keeps a_2 and a_3 fixed also keeps a_1 and a_4 fixed. That is, $\Phi(a_2, a_3) = \{\epsilon\}$, where ϵ is the identity, and of course $F(\{\epsilon\})$ is always the set of all the vertices.

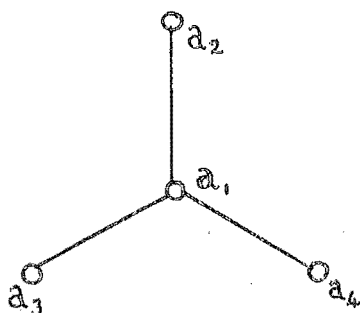


fig. 2.6.1.

Definition 2.6.3.

With the notation already used in this section we define cS , the automorphism-closure of S by

$$cS = F(\Phi(S)).$$

A set which is equal to its automorphism closure is called automorphism closed.

Theorem 2.6.3.

If $S_1 \supset S_2$ then $\Phi(S_1) \subset \Phi(S_2)$.

For if ϕ belongs to $\Phi(S_1)$ it keeps every vertex of S_1 fixed, and so a fortiori every vertex of S_2 . Hence ϕ belongs to $\Phi(S_2)$.

Theorem 2.6.4.

If Ψ_1 is a subgroup of Ψ_2 then $F(\Psi_1) \supset F(\Psi_2)$.

For if a is left fixed by every member of Ψ_2 it is certainly left fixed by every member of Ψ_1 .

These two theorems act as lemmas for the interesting theorem:

Theorem 2.6.5.

If Ψ is any subgroup of Γ , $F(\Psi)$ is automorphism closed.

For let ψ belong to Ψ . Then ψ keeps every vertex of F fixed, so $\psi \in \Phi(F(\Psi))$, the fixing subgroup of Ψ . Hence Ψ is contained in $\Phi(F(\Psi))$. Now this inclusion may be strict, but

as $\Psi \subset \Phi(F(\Psi))$ theorem 2.6.4 tells us that $F(\Phi(F(\Psi))) \subset F(\Psi)$. And by the definition of Φ it keeps every vertex of $F(\Psi)$ fixed, so $F(\Psi) \subset F(\Phi(F(\Psi)))$. These two sets are therefore equal.

As a corollary we have

Theorem 2.6.6.

The automorphism closure of any set of vertices is automorphism closed.

2.7 Given two vertices of a graph, there may or may not exist an automorphism mapping one of them onto the other.

Definition 2.7.1.

Two vertices a, b belong to the same transitivity class if and only if there is an automorphism ψ of the graph such that $\psi(a) = b$.

It may easily be established that this definition does in fact define disjoint equivalence classes.

Given any graph the discovery of the transitivity classes is an almost essential preliminary to the calculation of the automorphism group. In the later part of the chapter several criteria, none complete in themselves are described which can be used to distinguish vertices which cannot belong to the same transitivity class.

Theorem 2.7.1.

The number of vertices in any transitivity class divides the order of the group.

Let $\Phi(a)$ be the fixing subgroup of the vertex a in a graph and b some vertex in the same transitivity class as a . Then if ϕ belongs to $\Phi(a)$, $\phi(a) = a$, so that if ψ is an automorphism sending a to b , (and there necessarily is such an automorphism), $\psi\phi(a) = \psi(a) = b$. Hence every automorphism in the same left coset $\psi\Phi(a)$ sends a into b . On the other hand if ψ_1 and ψ_2 both send a into b , $\psi_1^{-1}\psi_2(a) = a$, so ψ_1 and ψ_2 belong to the same left coset relative to $\Phi(a)$. Hence the vertices in the transitivity class of a correspond precisely with the left cosets of $\Phi(a)$.

Theorem 2.7.2.

Two members of the same transitivity class have isomorphic fixing subgroups.

If a has fixing subgroup Φ and b belongs to the transitivity class of a , there is an automorphism ψ such that $\psi(a) = b$. Then if ϕ is a member of Φ , $\psi\phi\psi^{-1}(b) = b$, and the subgroup $\psi\Phi\psi^{-1}$ holds b fixed. Conversely if ω holds b fixed, $\psi^{-1}\omega\psi$ holds a fixed, so the fixing subgroup of b is precisely $\psi\Phi\psi^{-1}$ which is isomorphic with Φ .

There is of course no corresponding converse theorem. All the vertices of the graph in figure 2.7.1 have fixing

subgroups isomorphic to the cyclic group of order 2, but there are two transitivity classes.

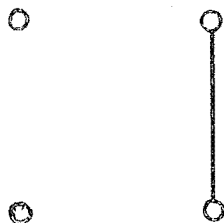


fig. 2.7.1.

2.8 As a first condition for two vertices to belong to the same transitivity class we prove the theorem:

Theorem 2.8.1.

If there is an automorphism ϕ of the graph G mapping the vertex a in G onto the vertex b , a and b have the same valency.

For let a be adjacent to the vertices a_1, a_2, \dots, a_n . Then $b = \phi(a)$ is adjacent to $\phi(a_1), \phi(a_2), \dots, \phi(a_n)$ and no other vertices. Similarly if a is adjacent to an infinite set of vertices, so is $\phi(a)$.

And as a second condition we note Theorem 2.5.3, which shows that vertices can only belong to the same transitivity class if they belong either to the same component or to isomorphic components.

For the rest of this section we concentrate on graphs which have only one transitivity class. Such a graph is

called transitive. From the above criteria all the vertices necessarily have the same valency and if the graph is not connected all its components are isomorphic. From Theorem 2.7.1 it follows that the number of vertices in a transitive graph divides the order of the group.

Since complementary graphs have the same automorphisms, the transitivity classes of complementary graphs consist of corresponding vertices. Hence the complement of a transitive graph is also transitive.

In contradistinction to the results of Chapter 3, not every abstract group can be represented as the automorphism group of a transitive graph. None of the cyclic groups of order greater than 2 can be so represented. For instance the cyclic group of order 3 could only be represented by a transitive graph with three vertices, since the order of the graph divides the order of the group. But no graph with three vertices has this group. The simplest graph with this group is shown in fig. 2.8.1. It has nine vertices, belonging to three transitivity classes.

Indeed Chao has shown (Ch1) that no transitive graph with more than two vertices has an abelian group.

The complete and null graphs are examples of transitive graphs, and so are the one-dimensional skeletons of the polygons and regular polyhedra, and their complements.

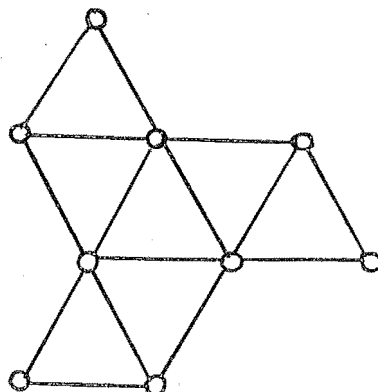


fig. 2.8.1.

2.9 There is in group theory the analogous concept of a transitive group. A permutation group is transitive if given any two members a, b of the permuted set there is a permutation π such that $\pi(a) = b$. The graph automorphism groups are permutation groups on the set of vertices, and the group of a transitive graph is a transitive permutation group.

But in group theory there is a further concept of multiplicity: a transitive group is k -ply transitive if there is a permutation in the group mapping any k distinct elements a_1, \dots, a_k onto any k distinct elements b_1, \dots, b_k . As it happens this idea does not carry over fruitfully to graphs. For let (a_1, a_2) be any two vertices and let there exist an automorphism for any given pair b_1, b_2 of vertices such that $\phi(a_1) = b_1$, $\phi(a_2) = b_2$. Then if (a_1, a_2) is an edge, so is (b_1, b_2) . So that if the graph is doubly or otherwise multiply transitive it is either a complete graph or a null

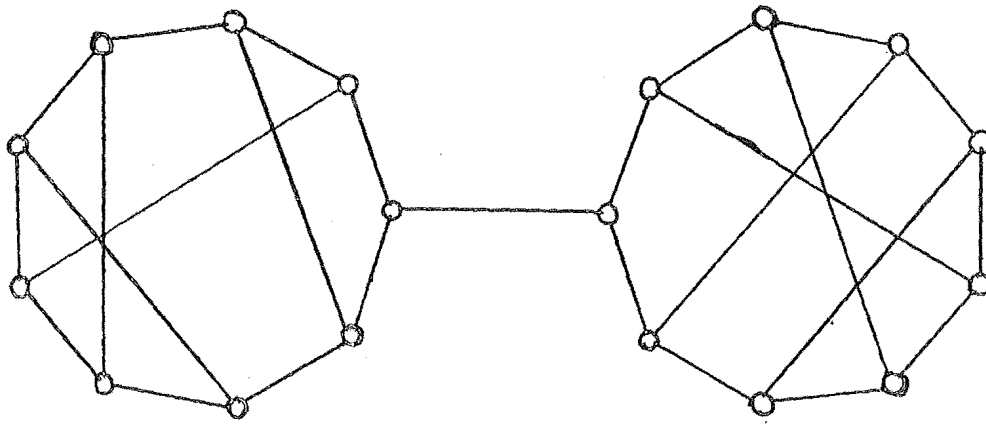


fig. 2.10.1

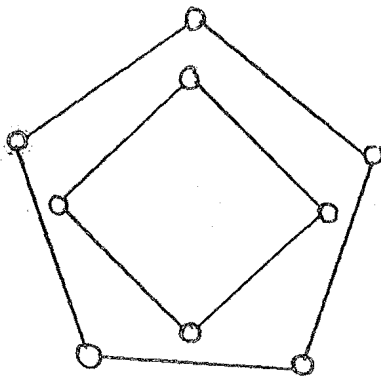
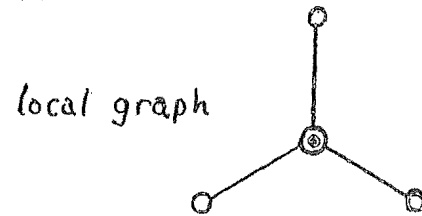


fig. 2.10.2

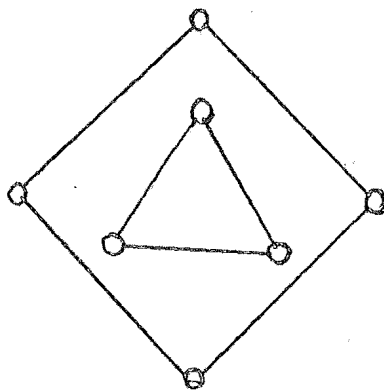


fig. 2.10.3

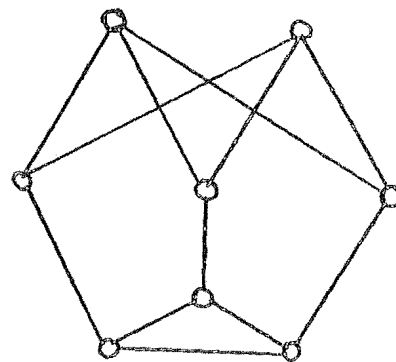


fig. 2.11.1

graph. (See for comparison Theorem 2.2.3.)

2.10 We showed in Theorem 2.8.1 that if two vertices belonged to the same transitivity class they had the same valency. This is not a sufficient condition, as the graph in fig. 2.10.1 shows. I therefore sought for a stronger condition, and one such is the local graph.

Definition 2.10.1.

Let a be a vertex in a graph G . Then $L(a)$, the local graph of a , is the subgraph of G whose vertices are a and the vertices adjacent to a .

Clearly any isomorphism ϕ mapping a onto a vertex of the same or another graph induces an isomorphism between the local graphs $L(a)$ and $L(\phi(a))$ in which a corresponds to a . Two members of the same transitivity class in any graph must therefore have isomorphic local graphs, in the restricted sense indicated above. The converse is not unfortunately the case.

We define a uniform graph to be a graph in which all the vertices have isomorphic local graphs. Every transitive graph is uniform, but a uniform graph may not be transitive. Indeed the graph of fig. 2.10.1 is uniform (local graph type at the right) but is also asymmetric.

Nor need the complement of a uniform graph be uniform: the graph of fig. 2.10.2 is uniform. Its complement is not.

Figure 2.10.3 is given to show that there are graphs in which every vertex has the same valency but which are not uniform. Such a graph is not therefore transitive.

2.11 The moment of a graph, defined in section 1.10, can also be used as a test for transitivity. As first defined it applies only to connected graphs, but the definition can easily be expanded to all graphs by taking the summation only over the vertices in the same component.

Theorem 2.11.1.

If a is mapped by an automorphism of G to $\phi(a)$, the moment about $\phi(a)$ is equal to the moment about a .

We proved in Theorem 2.5.1 that if there is a path from a to b there is a path of the same length from $\phi(a)$ to $\phi(b)$. Thus the distance $d(a,b)$, the length of the shortest path is also preserved, and as the vertices $b, \phi(b)$ are in one-to-one correspondence, with equal distances from a and $\phi(a)$, the moments about a and $\phi(a)$ are equal.

Definition 2.11.1.

A graph in which the moments about all vertices are equal is called equimomental.

From the above theorem all transitive graphs are equimomental; but that not all equimomental graphs are transitive will be seen from fig. 2.11.1. As it happens

the complement of this graph (which is not self-complementary) is also equimomental, but there are equimomental graphs whose complements do not have that property.

2.12 A final criterion was developed by Frucht (Fr 2). As he defined and used it it applied only to cubic graphs, that is, graphs in which every vertex is of valency 3, but it may be easily extended to apply to any graph.

In a cubic graph there are three edges, say e_1, e_2, e_3 incident with any given vertex p . Take any two of these, say e_1, e_2 . It may be that they do not belong together to any elementary circuit in the graph. If this is the case we write $c_{12} = \infty$. If on the other hand they belong to one or more elementary circuits, we give c_{12} the minimum of the lengths of all the circuits containing e_1 and e_2 . Similarly we define c_{23} from e_2 and e_3 and c_{13} from e_1 and e_3 . We then define the type of p to be the triple (k, l, m) whose elements are c_{12}, c_{23}, c_{13} arranged in non-descending order.

Infinity may certainly be included among the elements of the triple if the graph is infinite, but this can also be done for a finite graph. Indeed the central vertex of the graph in figure 2.12.1 has type (∞, ∞, ∞) .

This definition may easily be extended to vertices of all valencies higher than 1, by taking all the possible pairs of edges incident with the given vertex, and thus

obtaining $\frac{1}{2}v(v-1)$ entries in the type n -tuple, where v is the valency of the vertex.

That the type is not sufficient to define the transitivity classes, or the number of vertices of given type to differentiate between non-isomorphic graphs is seen from figures 2.12.2 and 2.12.3. Let these graphs be G and H . G has six vertices of type $(4,4,4)$, and H has eight vertices, also of this type. Each is transitive, and having different orders they are not isomorphic. But consider a graph K with two components, one isomorphic to G , the other to H . Then K is not transitive, though all its vertices are of the same type. Again let L_1 be a graph with four components, each isomorphic to G , and let L_2 be a graph with three components, each isomorphic to H . Then each graph has 24 vertices of type $(4,4,4)$ but clearly they are not isomorphic.

2.13 Finally we return from consideration of vertex invariants for isomorphism to look at a different aspect of fixed vertices and automorphism closure.

Definition 2.13.1.

A set D of vertices in a graph G is termed a determining set if the automorphism closure of D is the whole vertex set of the graph.

Alternatively this may be expressed by saying that D is a set of vertices such that only the identity keeps

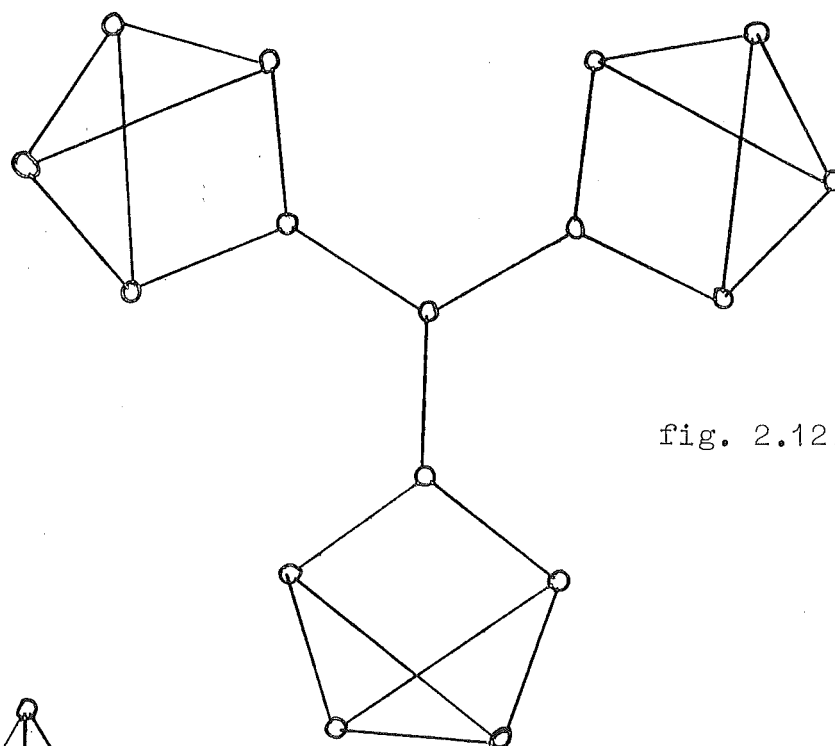


fig. 2.12.1

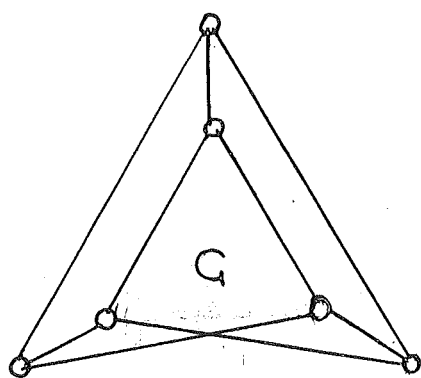


fig. 2.12.2

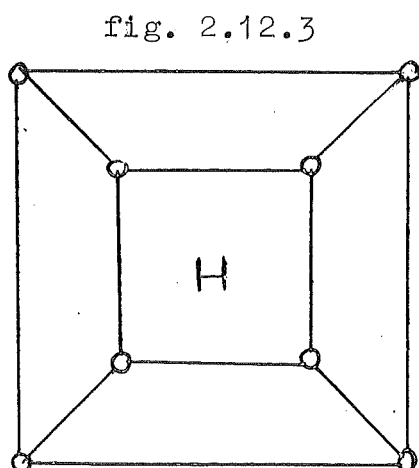


fig. 2.12.3

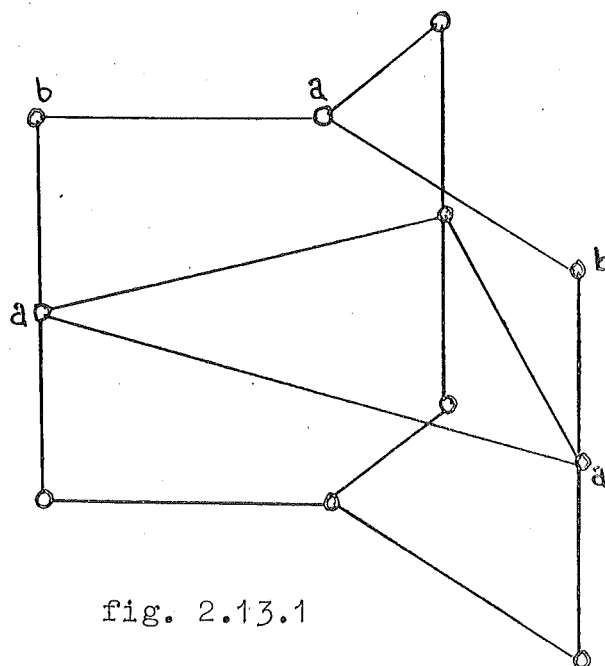


fig. 2.13.1

every member of D fixed. In the special case of an asymmetric graph every set, including the empty set, is a determining set.

Naturally the minimal determining sets, which contain no proper subsets which are determining sets, are the important ones.

It would be convenient if all minimal determining sets in a given graph contained the same number of members: unfortunately fig 2.13.1 shows that this is not the case. The three vertices marked a form one minimal determining set, the two vertices marked b form another.

Definition 2.13.2.

The degree of freedom of a graph is the largest order of any minimal determining set. It is taken as zero for asymmetric graphs. Some infinite graphs have infinite determining sets which have no minimal determining sets: the degree of freedom is then taken as infinite.

It may be wondered why we do not take as degree of freedom the least order, for then we need not explicitly mention minimal determining sets. The answer is that this definition is forced upon us by the use made of this concept in Chapter 5: the other definition would also yield a weaker theorem below.

Theorem 2.13.1.

Let G be a graph with a group Γ of finite order g , and having degree of freedom r . Then g has a factorisation into at least r factors each greater than 1.

For let $D = \{a_1, a_2, \dots, a_r\}$ be a minimal determining set. Then none of these vertices is fixed and for each s , ($1 \leq s \leq r$) there is defined a fixing subgroup Φ_s which holds fixed all the vertices a_1, a_2, \dots, a_s . Now the subgroups are nested

$$\Gamma \supset \Phi_1 \supset \Phi_2 \supset \dots \supset \Phi_r = \{e\}.$$

Since the set is minimal we have that for each s ($1 \leq s \leq r$), a_s does not belong to the automorphism closure of the set $\{a_1, \dots, a_{s-1}\}$. Hence the inclusions above are strict. Hence the order has one factor for each inclusion, r factors in all.

Applying the theorem in reverse gives the maximum possible degree of freedom of any graph with a given finite order of group.

2.14 Finally in this chapter it is necessary to distinguish my own contributions. The discussion of fixing subgraphs etc., automorphism-closure (although the idea has previously been applied to groups) and degree of freedom is mine. Transitivity classes have been long defined, but the proof of Theorem 2.7.1, and Theorem 2.7.2 are my own. The local graph and moments are also my own contribution.

CHAPTER III

FRUCHT'S PROBLEM

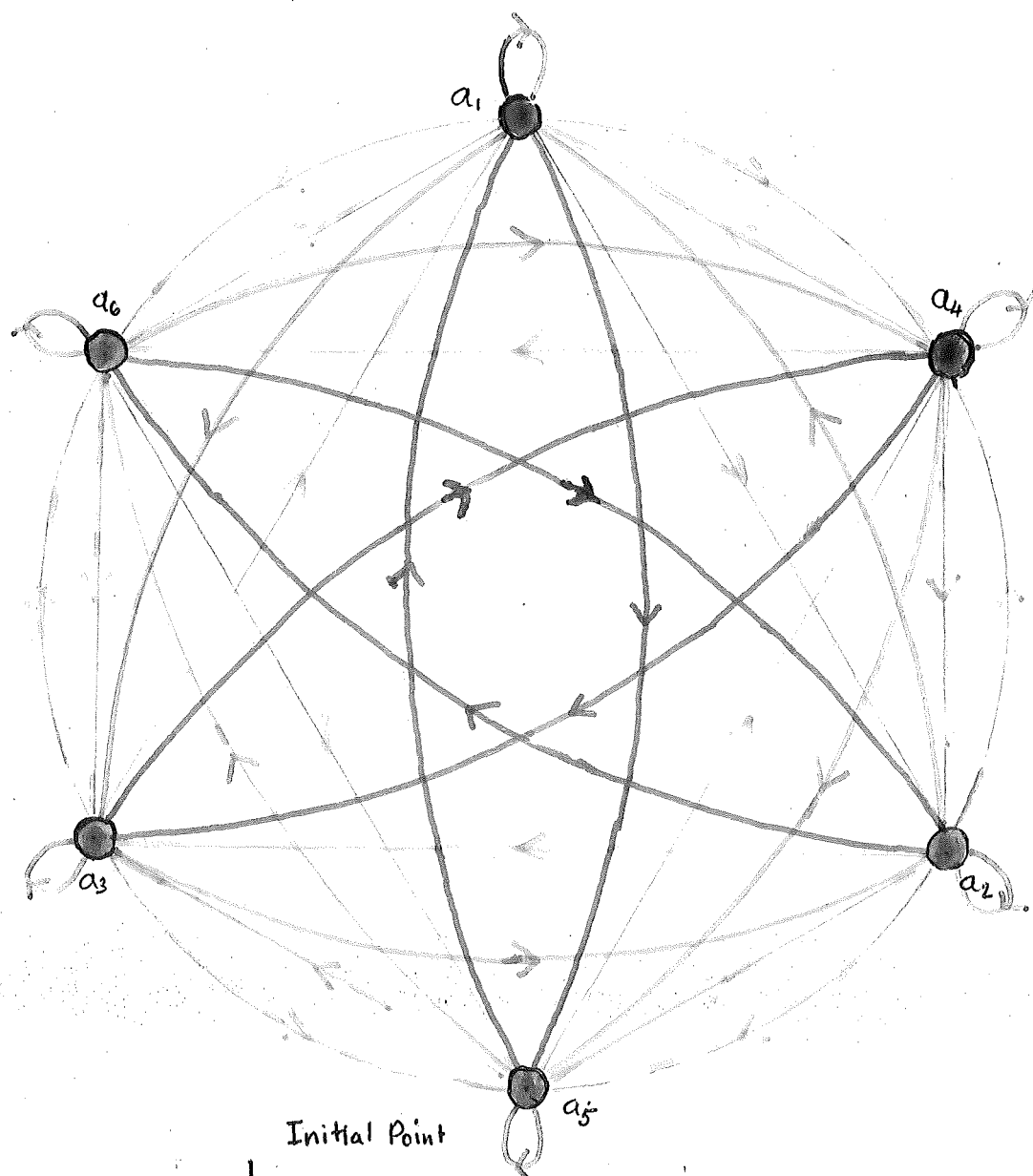
3.1 The automorphisms of a graph were first considered by Jordan (Jo 1) in 1869, but without the modern terminology. König in his book (Ko 1) refers briefly to the group of a graph and posed the problem whether given any abstract group it was possible to construct a graph with group isomorphic to the given group. This problem was answered in the affirmative for finite groups by Frucht (Fr 4) in 1938, and I have given his name to the problem. This chapter is devoted to an account of this first solution, a brief summary of more recent work, and related problems. Consequently relatively little of it is my own. Such sections will be pointed out as we meet them.

3.2 To give Frucht's 1938 solution we must first go back to Cayley, who in several papers, but principally (Ca 9) developed what has come to be known as the 'Cayley colour group' method of representing a group.

The colour group resembles a graph in many of its features. It consists of a collection of vertices, each corresponding to a member of the group. Let ϕ, ψ, ω be members of the group Γ which is being represented. Then there is a directed line between each pair of vertices:

again the lines out of a vertex are made to correspond with the various members of the group. There is a loop from each vertex to itself, which always corresponds to the identity. The correspondences between vertices and lines are related by the rule that passing from the vertex corresponding to ϕ along the line corresponding to ψ we reach the vertex corresponding to $\psi\phi$. In other words, the line from the vertex corresponding to ϕ to the vertex corresponding to ω corresponds to $\omega\phi^{-1}$. As a consequence of this rule the line from the vertex of ω to the vertex of ϕ corresponds to $\phi\omega^{-1}$: lines in opposite directions between the same pair of vertices belong to inverses. A further correspondence is then also made for diagrammatic purposes: a colour or other pictorial mode of differentiation is made to correspond to each group member. The result is then of the kind shown in fig. 3.2.1, which is drawn for the symmetric group on three symbols. The directed lines only are coloured.

Several features may be noted, consequent upon the manner of definition. Concentrating on lines of one colour, they form disjoining cycles, all of the same length: the number of lines in each cycle is the order of the corresponding group element. Every vertex lies on precisely one such cycle. (If the group has elements of infinite order, the cycles are of infinite length, and become open paths rather than cycles.) Cycles of length 2 can occur, the line from vertex a to vertex b being of the same



Initial Point		γ_1	γ_2	γ_3	γ_4	γ_5	γ_6
Arc	$\downarrow \gamma_1$	γ_1	γ_2	γ_3	γ_4	γ_5	γ_6
	γ_2	γ_2	γ_3	γ_1	γ_6	γ_6	γ_5
	$\downarrow \gamma_3$	γ_3	γ_1	γ_2	γ_5	γ_6	γ_4
	$\downarrow \gamma_4$	γ_4	γ_5	γ_6	γ_1	γ_2	γ_3
	$\downarrow \gamma_5$	γ_5	γ_6	γ_4	γ_3	γ_1	γ_2
	$\downarrow \gamma_6$	γ_6	γ_4	γ_5	γ_2	γ_3	γ_1

Entry
is
endpoint

fig. 3.2.1

colour as the line from b to a . The loops corresponding to the identity are cycles of length 1. In each case there is a coherent orientation to each cycle - each line being traversed in the direction of its definition. We also find that if any path is traced in the forward direction along lines corresponding to $\phi_1, \phi_2, \dots, \phi_r$ in that order, the initial and final points are joined by a line corresponding to $\phi_r \phi_{r-1} \dots \phi_2 \phi_1$.

All the vertices are alike in that there is one line of each colour into, and one line of each colour out of, each vertex. We consider the possible automorphisms of this figure. We will consider as an automorphism any permutation of the vertices which preserves the colour of the lines in the sense that if π is an automorphism, the colour of the line from $\pi(a)$ to $\pi(b)$ is always the same as the colour of the line from a to b . Consider for instance the automorphism of the figure 3.2.1 defined by

$$\begin{aligned}\pi(a_1) &= a_4, \pi(a_2) = a_6, \pi(a_3) = a_5, \pi(a_4) = a_1, \pi(a_5) = a_3, \\ \pi(a_6) &= a_2.\end{aligned}$$

The line from a_2 to a_3 is orange. The line from $a_6 (= \pi(a_2))$ to $a_5 (= \pi(a_3))$ is also orange.

These automorphisms define a group, as may be shown directly from the mode of definition. Let this group be Π .

Theorem 3.2.1.

The automorphism group Π of a Cayley colour group is isomorphic to the given group Γ .

Let a_i, a_j be two vertices of the figure, and π an automorphism of the figure. Now a_i and a_j are connected by a line from a_i to a_j corresponding to $\gamma_j \gamma_i^{-1}$, and the line from $\pi(a_i)$ to $\pi(a_j)$ corresponds to $\pi(\gamma_j)[\pi(\gamma_i)]^{-1}$. (There is a slight abuse of notation here: π is used for both the permutation of the vertices and a corresponding mapping of the elements of Γ .) Since these two lines must correspond to the same element of Γ ,

$$\begin{aligned}\gamma_j \gamma_i^{-1} &= \pi(\gamma_j)[\pi(\gamma_i)]^{-1}, \text{ so that} \\ \pi(\gamma_j) &= \gamma_j \gamma_i^{-1} \pi(\gamma_i).\end{aligned}$$

Now let us fix i , so that γ_i is the identity ε of the group Γ . Then $\pi(\gamma_j) = \gamma_j \pi(\varepsilon)$. There is then seen to be precisely one automorphism π_r of the figure with the property $\pi_r(\varepsilon) = \gamma_r$. The groups Π and Γ thus have the same order.

We define the product in Π in the usual way,

$$\pi_s \pi_r(\gamma_j) = \pi_s(\pi_r(\gamma_j))$$

Since we know that $\pi_r(\gamma_j) = \gamma_j \gamma_r$,

$$\pi_s \pi_r(\gamma_j) = \pi_s(\gamma_j \gamma_r) = (\gamma_j \gamma_r) \gamma_s = \gamma_j (\gamma_r \gamma_s).$$

The simple correspondence $\pi_r \rightarrow \gamma_r$ fails to be an isomorphism since the order of the terms is reversed, $\pi_s \pi_r$

corresponding to $\gamma_r \gamma_s$. The order may be restored by defining the isomorphism ϕ of Π onto Γ by $\phi(\pi_r) = \gamma_r^{-1}$. (Alternatively the correspondence of π_r to γ_r could be obtained by defining $\pi_s \pi_r(\gamma_j) = \pi_r(\pi_s(\gamma_j))$ or in the original definition of the correspondence of the lines and vertices defining the line from a_i to a_j to correspond to $\gamma_i^{-1} \gamma_j$, so that passing from a_i along an edge corresponding to γ_k we reach the vertex corresponding to $\gamma_i \gamma_k$).

3.3 Frucht's first solution (Fr 4) is obtained in a simple fashion from the Cayley colour group. All that is necessary is to replace each coloured directed line by a suitable subgraph. Such a subgraph must be different from all the others, corresponding to the other colours, and must have its end-points distinguished. The loops corresponding to the identity are a special case: they are simply omitted. Care must also be taken that every isomorphism of the resulting graph must map the set of original vertices onto itself: these vertices must precisely form one transitivity class. Every other transitivity class must contain just one vertex from each subgraph replacing the edges of a particular colour. The graph must in fact have one degree of freedom, fixing the image of any vertex fixes the images of all the vertices. Frucht's version of the subgraphs to replace the lines corresponding to $\gamma_2, \gamma_3, \dots, \gamma_g$ (with γ_1 the identity) are shown in fig. 3.3.1. In fig. 3.3.2 is a more efficient

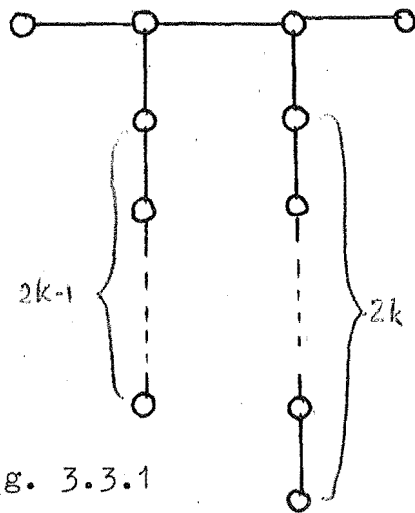


fig. 3.3.1

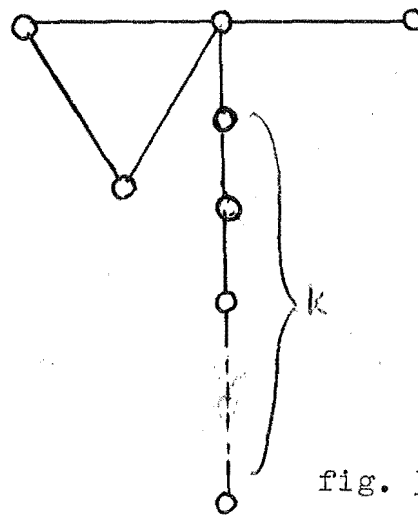
 γ_k 

fig. 3.3.2

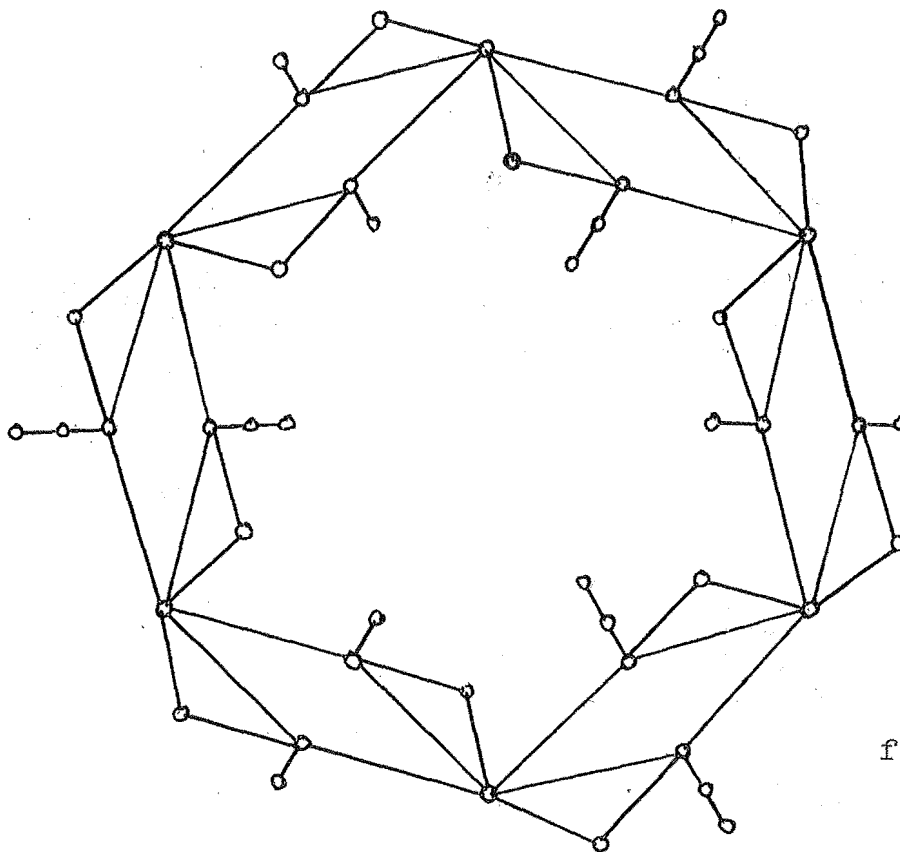


fig. 3.3.3

version.

Using Frucht's subgraphs a group of order g requires $g^2(2g-1)$ vertices; the more efficient method reduces this to $\frac{1}{2}g^2(g+3)$. A further reduction may be made since the Cayley colour group can itself be simplified by retaining only the lines belonging to a set of generators. Translating this in the same way gives a graph with $\frac{1}{2}g(h^2+5h+2)$ vertices according to the more efficient version, where h is the number of generators.

Fig. 3.3.3 shows the graph obtained for S_3 using γ_4 and γ_6 as generators. Graphs of this type will be referred to as Frucht graphs.

In a later paper (Fr 2), Frucht produced a more economical version, also derived from the Cayley colour group but quite different in appearance. For non-cyclic groups the number of vertices necessary is reduced to $2gh$, (g and h as defined above), and for cyclic groups $3g$ vertices are necessary. These graphs have the added property that each vertex has valency 3.

Even this is not the best that can be obtained in all cases. For instance the four-group $(C_2 \otimes C_2)$ has $g = 4$ and $h = 2$, giving 16 for the number of vertices in its graph by Frucht's second solution. But the graph with four vertices and only one edge has this group.

3.4 In (Sa 1) Sabidussi extended Frucht's work to show

that not only could a graph be found with group isomorphic to any given group, but that any one of the following properties could be required and that there would still be an infinity of such graphs which were fixed point free:

- (1) the connectivity is any integer greater than or equal to 1.
- (2) the chromatic number is any integer greater than or equal to 2.
- (3) all the vertices have the same valency, greater than or equal to 3.
- (4) the graph is spanned by a graph homeomorphic to a given connected graph.

(Some of these terms are not used elsewhere in this thesis and have not therefore been defined.)

These results can be expressed in the following way:

Definition 3.4.1.

Two graphs are isosymmetric if their automorphism groups are isomorphic.

The set of all graphs is then divided into isosymmetry classes. Sabidussi's results show that every isosymmetry class contains an infinity of fixed-point-free graphs with the above properties. Frucht's first solution also shows that there is at least one fixed point free graph with one degree of freedom in each isosymmetry class, and it is easy to modify the solution to provide an infinity of such

graphs. We have shown (section 2.8) that not every class contains a transitive graph. Nor, as we shall see in a later section, does every class contain a tree. Every class does contain a connected graph, since the graphs constructed in this and the previous section are connected. All classes contain non-connected graphs; take any connected graph with the given group (except the trivial graph for the trivial group) and construct a graph with two components, one isomorphic to the above graph and the other an isolated vertex.

3.5 Just before Frucht's first solution was produced, Polya had solved the problem when the graphs are restricted to trees. His approach is based on Jordan's earlier work on the automorphisms of a graph (Jo 1), and Polya's statement of the result (Po 2 p 208) may be translated:

"For trees the Jordan method gives a more concrete result than for other graphs: with each tree we can associate the natural numbers m_1, m_2, \dots, m_r , $r \geq 1$, $m_1 < m_2 < \dots < m_r$, $m_1 + m_2 + \dots + m_r \leq n$ (the order of the tree) so that the automorphism group of the tree can be represented by the symmetric groups $S_{m_1}, S_{m_2}, \dots, S_{m_r}$ with some ordering of the operations direct product and wreath product $G(H)$."

This statement is open to misinterpretation: it may be taken to imply that the group $S_m \otimes S_m$ is not allowed, whereas the same integer may in fact be repeated any number

of times. Also it does not sufficiently restrict the type of wreath product which can be allowed: only $S_m(H)$ is possible.

What follows is my own solution, which also brings out certain points not mentioned above.

Since we will be concerned only with the wreath product $S_n(\Gamma)$ we will use the shorter notation Γ^n introduced in section 2.5. We notice also that if I is the trivial group, $S_n(I)$ is isomorphic to S_n , enabling us to use I^n as an alternative notation to S_n . We then state the main theorem of this section:

Theorem 3.5.1.

Any tree T has an automorphism group which may be obtained from the trivial group by the operations of wreath product Φ^n and direct product $\Phi \otimes \Psi$ in some order. Conversely given such a group there is a tree with this group.

We begin by considering rooted trees.

Definition 3.5.1.

A rooted tree is any tree in which one vertex (the root) has been differentiated from the others.

The automorphisms of a rooted tree are those automorphisms of the corresponding tree which keep the root fixed. They therefore form precisely the fixing subgroup corresponding to the root of the group of all automorphisms

of the tree.

First suppose that the rooted tree consists simply of the root and n vertices each adjacent only to the root (fig. 3.5.1). The group of such a rooted tree is clearly S_n, I^n in the notation we are using here.

Now let R be a rooted tree with root r and vertices a_1, \dots, a_m adjacent to r . Each of these defines a subtree R_i , rooted at a_i , defined by those vertices which are reached from r by a path through a_i . Since the tree contains no elementary circuits and is connected each vertex of R , apart from r , belongs to precisely one R_i . Some of these R_i may be trivial, consisting only of the root a_i , see fig. 3.5.2. Let the group of R_i (as a rooted tree) be Γ_i .

Theorem 3.5.2.

Every automorphism of the rooted tree R maps each vertex adjacent to r to another vertex adjacent to r . It therefore defines a permutation on the R_i .

It is necessary in proof only to remark that as r is kept fixed, any vertex adjacent to r must be mapped into a vertex adjacent to r , and that if b and b' are two vertices there is a path from one to the other not through r if and only if they belong to the same sub tree R_i . Hence there is a path between their images not through r if and only if the images belong to the same subgraph.

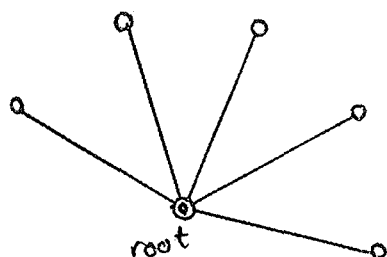


fig. 3.5.1

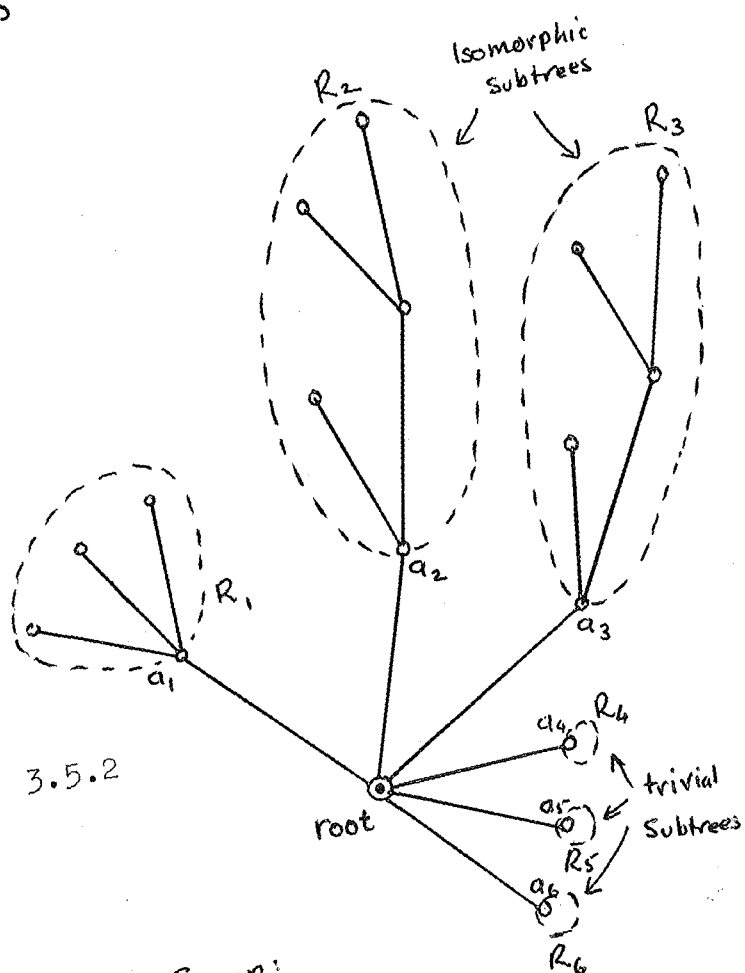


fig. 3.5.2

Group:
 $I^3 \otimes (I^2)^2 \otimes I^3$

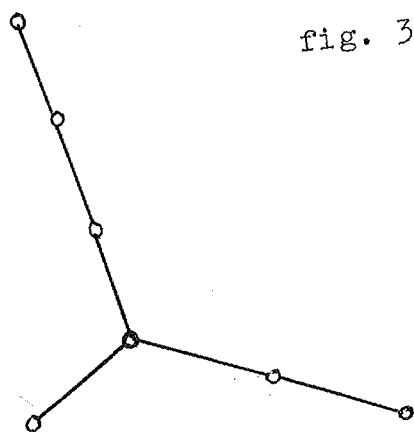


fig. 3.5.3

Theorem 3.5.3.

Let R_i, R_j be two subtrees as defined above. Then there is an automorphism of R which interchanges R_i and R_j if and only if R_i and R_j are isomorphic.

This theorem is analogous to Theorem 2.5.3, and the proof is along the same lines.

The remainder of the theory, leading to the group of R in terms of the groups Γ_i also closely parallels the later discussion of section 2.5. If there are m_1 trees isomorphic to R_1 (including R_1) the group of this collection of subtrees is $\Gamma_1^{m_1}$. We state the result.

Theorem 3.5.4.

If $\Gamma_1, \Gamma_2, \dots, \Gamma_s$ are the groups of the non-isomorphic subtrees, and if each occurs with frequency m_1, \dots, m_s , the group Γ of R is

$$\Gamma_1^{m_1} \otimes \Gamma_2^{m_2} \otimes \dots \otimes \Gamma_s^{m_s}.$$

We note explicitly that non-isomorphic subtrees can have isomorphic groups.

As each subtree can itself be broken down into subtrees in the same manner unless it is trivial, each non-trivial Γ_i can be split into a product after the above manner. Continuing the process the subtrees are continually decreased so that eventually all the subtrees are trivial. We thus obtain the theorem:

Theorem 3.5.5.

Any rooted tree R has an automorphism group which may be obtained from the trivial group by the operations of wreath product Φ^n and direct product $\Phi \otimes \Psi$ in some order. Conversely any such group can be represented as the automorphism group of some rooted tree.

The converse is easily shown, since we know how to construct a graph with group I^n and also how to produce the products Φ^n and $\Phi \otimes \Psi$. We can produce any number of rooted trees with a given group: let R be any rooted tree with group Γ and root r . Then we introduce a new vertex r' and a new edge (r, r') to create a new tree R' rooted at r' , and having the same group Γ . We can therefore form arbitrarily long direct products

$$\Gamma \otimes \Gamma \otimes \Gamma \otimes \Gamma \dots \otimes \Gamma.$$

The group of a tree now follows quite simply. Suppose a tree has at least one fixed vertex. Then the automorphisms of this tree are identical with the automorphisms of the rooted tree obtained from it by taking the fixed vertex as a root. Theorem 3.5.1 is therefore proved in this case.

In Sections 1.9 and 1.10 we mentioned three ways in which centres or bicentres may be defined for trees. As each definition involves only features which are invariant under isomorphism, any vertex which is a centre by any one

of these definitions is a fixed vertex. Any pair which are bicentres are either both fixed or form together a whole transitivity class: neither can be mapped to any other vertex in the graph. We have in fact the theorem:

Theorem 3.5.6.

Every tree without fixed vertices contains a pair of vertices which are interchanged or left fixed by every automorphism of the tree.

Let these two vertices be a and b . Then the edge (a,b) is always mapped into itself (perhaps with its ends interchanged) by every automorphism of the tree. Suppose we introduce into this edge a new vertex c . Then the automorphisms of the graph obtained correspond precisely with those of the old, and the new graph is also a tree. Further c is always mapped into itself. Formally:

Theorem 3.5.7.

Every tree is isosymmetric to a tree with a fixed vertex.

Since theorem 3.5.1 is already proved for trees with a fixed point, this completes the proof.

As to the converse result, that given any group we can find a tree with that group, we already know that we can find a rooted tree with that group. All that remains is to construct a rooted tree in which the root is fixed

in the corresponding tree. This can be done quite simply where the rooted tree is non-trivial. Let the root be r , and let the rooted tree have order n . Then we add to the rooted tree n new vertices b_1, b_2, \dots, b_n and edges $(r, b_1), (b_1, b_2), (b_2, b_3), \dots, (b_k, b_{k+1}), \dots, (b_{n-1}, b_n)$. This subgraph is clearly asymmetric and has too many vertices to be isomorphic to any of the other rooted subtrees. In the corresponding tree it can be shown that r is fixed.

This method may not work if the rooted tree is trivial, but there is in any case the asymmetric tree in fig. 3.5.3.

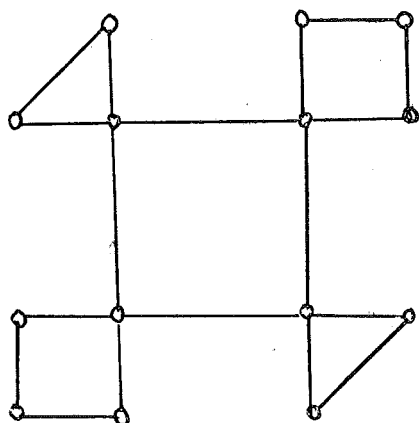
Combining Theorem 3.5.1 with the treatment in section 2.5, the collection of possible groups of a forest (a graph all of whose components are trees) is of the same kind as the groups of trees.

Since not every group is of this kind - the cyclic group of order 3 is not - not every group is the group of some tree.

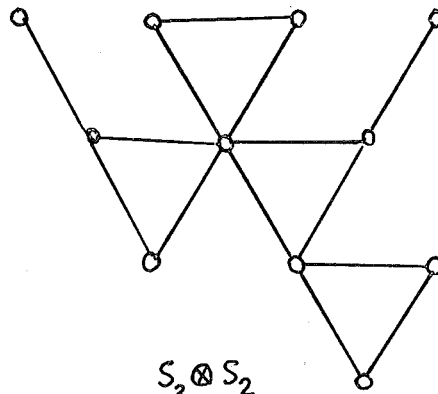
3.6 A 'cactus' or 'Husimi tree' is a graph in which each edge features in at most one elementary circuit. These graphs therefore include trees, and each group which can be represented as the group of a tree can be represented as the group of a cactus.

It might be thought that the widening of the class

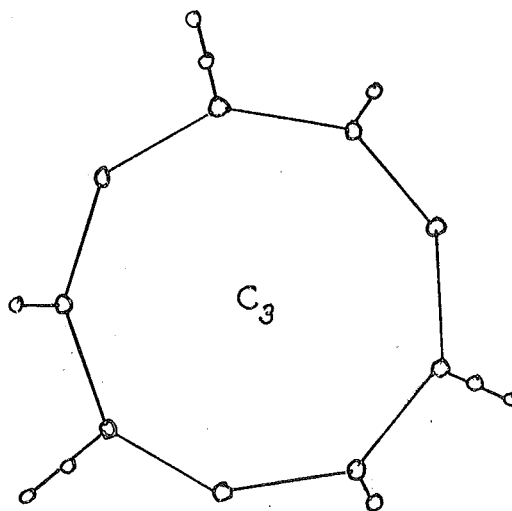
fig. 3.6.1 Some cacti and their groups



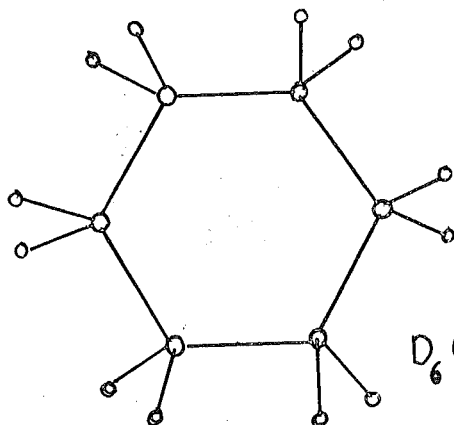
$$S_2(S_2) \otimes S_2(S_2)$$



$$S_2 \otimes S_2$$



$$C_3$$



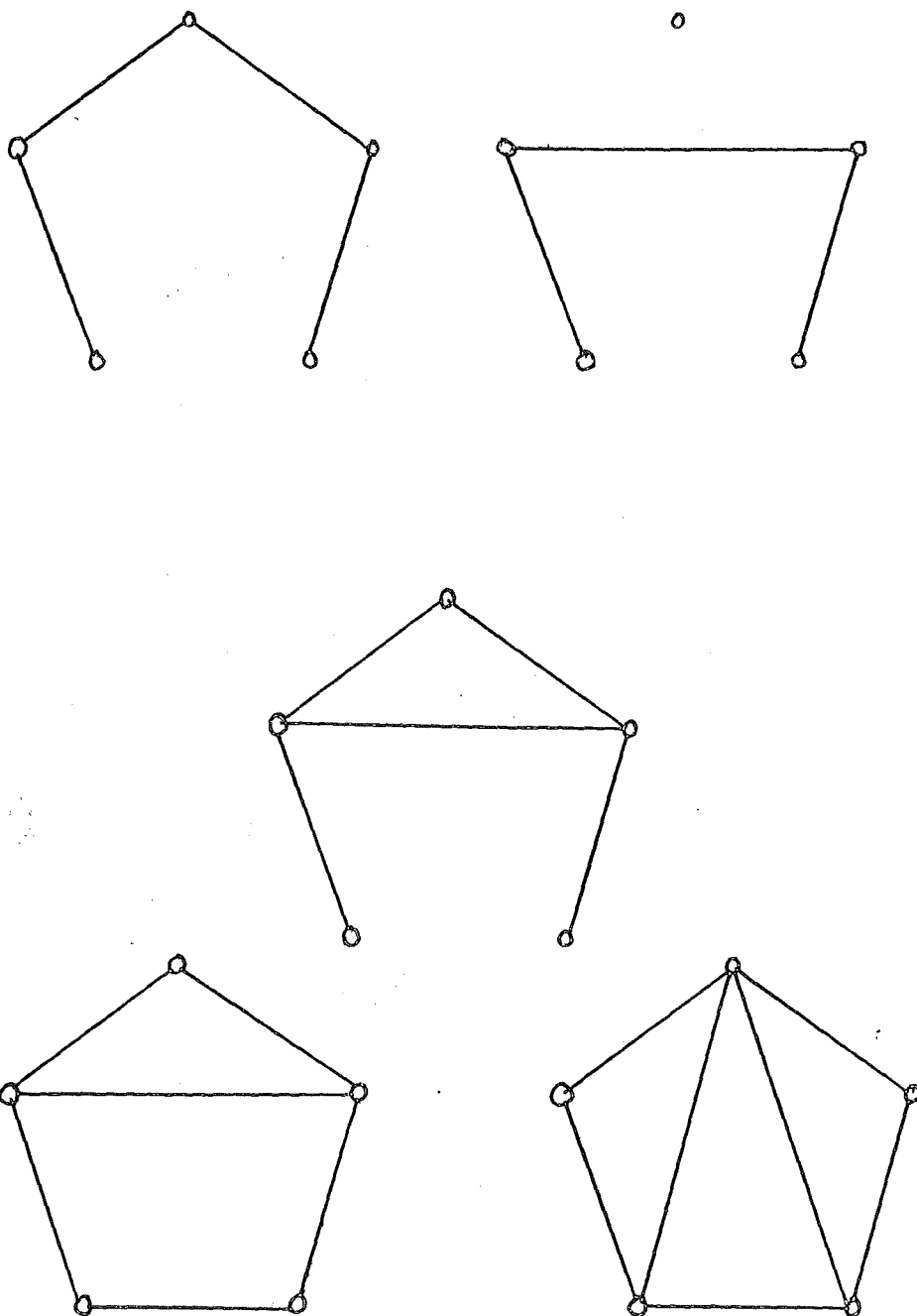
$$D_6(S_2)$$

would greatly widen the class of groups which can be represented, but the increase is in fact quite modest. Rotations of at most one elementary circuit are possible, so that groups of the form $D_n(\Gamma)$ and $C_n(\Gamma)$ (dihedral and cyclic products) are obtained, where Γ is a tree group. Some examples are given in fig. 3.6.1. Although the cyclic group of order 3 can now be represented, there remain many groups which cannot.

3.7 So far we have considered the problem of constructing a graph with automorphism group isomorphic to a given abstract group. A related problem is the representation of a permutation group by a graph, where the permuted objects become the vertices of the graph. As remarked earlier, there is no graph on three vertices whose group is the cyclic group of order 3. Hence the group consisting of the permutations $\{(1), (123), (132)\}$ does not have a representation and the problem does not always have a solution.

The graph whose automorphism group is a given permutation group must necessarily have order the same as the degree of the permutation group. As the groups of a given order are certainly finite in number, each permutation group has at most a finite number of representations. Since the automorphisms of a graph and its complement coincide, the representations normally occur in complementary pairs. But this does not happen when the graph is self-

fig. 3.7.1



complementary. A group may have more than one pair of representations, and also a given graph may often be numbered in several ways to give representations of the same group. This is illustrated in fig. 3.7.1, where the group $\{(1), (1)(23)(45)\}$ is represented in five ways. There are two complementary pairs, and one self-complementary graph. In addition each can be numbered in four distinct ways, so that we may consider there to be twenty representations.

We will not attempt to devise any necessary and sufficient set of conditions for the existence of the representation of a permutation group, but some conditions can be stated simply.

As we showed in section 2.9 a multiply-transitive graph is either null or complete, so that a group of transitivity multiplicity greater than 1 cannot be represented unless it is the symmetric group. As a corollary to this, the alternating groups A_n are also multiply transitive and cannot be represented for n greater than 2.

Nor can the cyclic permutation groups be represented. Suppose the cyclic group C_n is generated by the permutation $(1\ 2\ 3\ \dots\ n)$, and that it is represented by a graph with vertices a_1, \dots, a_n , so that for each r , $1 \leq r \leq n$, a_r corresponds to r . Then the generating permutation defines the automorphism $\phi(a_r) = a_{r+1}$ on the graph. (For simplicity we here take the suffices modulo n). Now if

(a_p, a_q) is an edge, so is (a_{p+r}, a_{q+r}) for each integer r . In particular, if $r = n - p - q$, (a_{n-p}, a_{n-q}) is an edge. Hence the transformation $\psi(a_r) = a_{n-r}$ is an automorphism. But ψ is not a power of ϕ , except when $n = 2$ and ψ is then the identity. So the graph does not have the given permutation group as automorphism group. Thus for $n > 2$, C_n is not representable.

On the other hand the dihedral groups D_n are all representable, by a simple polygon with n vertices.

Kagno (Ka 4,5) was the first to publish in this field and obtained the above result. He also considered all permutation groups of degree less than 7 and found which of them belong to graphs in the restricted family of graphs he used: every vertex had valency at least 3.

CHAPTER IV

SYMMETRIC EMBEDDINGS

4.1 The remaining chapters of this thesis will discuss various embeddings of a graph into the real coordinate spaces R^n . The class of embeddings considered (as defined below) is more restricted than the general topological embeddings, in that the edges must be represented not simply as arcs but as straight line segments. The embeddings are in fact the 'straight' embeddings of a one-dimensional polytope.

Points in R^n will be denoted by their position vector, which will in turn be indicated by a lower case Roman letter with a straight underline.

Definition 4.1.1.

Given a graph G , an embedding G' of G in the real coordinate space R^n consists of a set of points $\{\underline{a}_i\}$ in R^n corresponding to the vertices $\{a_i\}$ in G , with the edges of the graph represented by straight line segments between the points representing the vertices. Furthermore the following conditions hold:

If a point \underline{p} in R^n represents a vertex of the graph it possesses a spherical neighbourhood which meets no other

point representing a vertex in G , and only those line segments representing edges in G incident with the vertex represented by p .

If p is a point in one of the line segments in G' , p has a spherical neighbourhood which contains no point representing a vertex of G and meets no other line segment in G' .

If p is not a point of G' , p has a spherical neighbourhood which contains no points of G' .

Theorem 4.1.1.

This definition implies directly the following results:

- (1) The set of points representing the vertices of G has no limit points.
- (2) No two line segments intersect other than at a point representing a vertex which is incident with each of them.
- (3) No vertex is represented by a point which lies on a line segment representing an edge which is not incident with it.
- (4) No graph containing a vertex with an infinite valency can be embedded in any R^n .
- (5) No graph with an uncountable number of vertices can be embedded in any R^n . (Since each R^n is completely separable and (HY1, p 65), in any completely separable space every uncountable subset contains

uncountably many limit points of itself).

Theorem 4.1.2.

Any graph satisfying the conditions (4) and (5) of Theorem 4.1.1 can be embedded in R^3 .

For each such graph is a 1-dimensional polytope and there is a general theorem that each n -dimensional polytope can be embedded (in our sense) in R^{2n+1} . (See for instance (HY4, p 215)).

It may be useful to have an explicit embedding which will work for any graph. If we make $\underline{a}_m = (m, m^2, m^3)$ for each positive integer m , (the graph must be of countable order in view of theorem 4.1.1 (5)) then no four vertices are coplanar. (Here and from now on we will often refer to the points of G' which represent vertices as the vertices of G' .) Hence the edges cannot meet improperly, and the set of vertices has no limit points, and the conditions in the definition of an embedding may be verified. It will of course also be necessary that each vertex is of finite valency.

We will assume for the rest of this thesis that the conditions (4) and (5) of theorem 4.1.1 hold without explicitly stating this.

4.2 A complete classification of the graphs which can be embedded in R^1 is easily accomplished. No vertex can have valency greater than 2, and the graph must contain no

circuits, and not more than 2 infinite components. Possible finite components are isolated vertices and components with vertices a_1, \dots, a_n and edges (a_r, a_{r+1}) for $r = 1, \dots, n-1$. There are two possible types of infinite component:

- I. Vertices $a_1, a_2, \dots, a_n, \dots$ and edges (a_r, a_{r+1}) for all $r \geq 1$;
- II. Vertices a_r for and edges (a_r, a_{r+1}) for all integers r .

There are then restrictions on the numbers of components. If there are no infinite components, or only one of type I, there may be an infinity of finite components. If there are two infinite components they must both be of type I, and there may also be a finite number of finite components. If there is an infinite component of type II there can be no other components. There cannot be more than two infinite components.

4.3 The graphs which can be embedded in R^2 have been studied in great detail, partly under the influence of the four-colour problem. These are the graphs termed planar. Although the line segments are not normally required to be straight our extra condition does not in fact disqualify any graphs.

We simply state the theorem, due to Kuratowski:

Theorem 4.3.1.

The necessary and sufficient condition for a graph

G to be planar is that it should possess no partial subgraph homeomorphic to the graphs in fig. 4.3.1 (the complete graph on 5 vertices) or fig. 4.3.2.

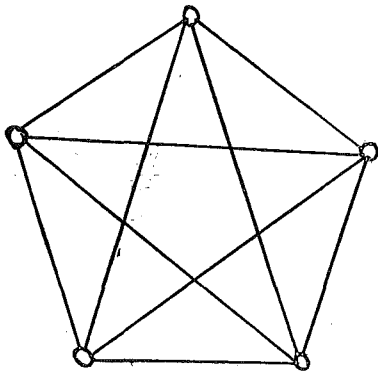


fig. 4.3.1.

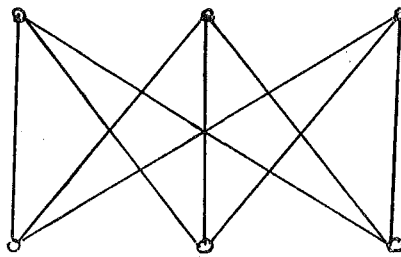


fig. 4.3.2.

The proof may be found in (Be 1 pp 211-213).

4.6 We next consider groups of one-to-one functions mapping R^n onto itself. There are many such groups: we shall be most interested in

- (a) All one-to-one functions of R^n onto itself
- (b) All homeomorphisms of R^n onto itself
- (c) All invertible affine transformations
- (d) All invertible linear transformations
- (e) All orthogonal affine transformations
- (f) All orthogonal linear transformations

- (g) All invertible piecewise-affine transformations
(defined in section 4.10).

Groups (b) and (g) are the main classes for the rest of this chapter, (d) and (f) for chapter 5 and (c) and (e) for chapter 6.

4.5 Let ϕ be a one-to-one function of R^n onto itself, and G' an embedding of a graph G in R^n . Suppose that ϕ permutes the points of G' representing the vertices of G , and further maps the line segments of G' onto line segments; $(\underline{a}_i, \underline{a}_j)$ onto $(\phi(\underline{a}_i), \phi(\underline{a}_j))$. Then ϕ may be said to induce a corresponding automorphism of G .

Conversely, given an automorphism ϕ' of G , this will define many one-to-one functions of G' onto itself, which may sometimes be extended to the whole of R^n to a member of some group of one-to-one functions of R^n onto itself.

Definition 4.5.1.

Let Φ be a group of one-to-one functions mapping R^n onto itself. Then an embedding G' of a graph G in R^n is called a Φ -symmetric embedding if every automorphism of G can be represented, not necessarily uniquely, as the function induced in G' by some member of Φ .

Theorem 4.5.1.

The members of Φ which induce automorphisms of G

in G' form a subgroup Ψ of Φ .

The proof is immediate.

Theorem 4.5.2.

Let G' be a Φ -symmetric embedding of a graph G in R^n . Let Ω be the subgroup of Φ which induces the identity automorphism in G . Then the automorphism group Γ of G is isomorphic to the factor group Ψ/Ω , where Ψ is the subgroup of Φ consisting of functions generating automorphisms in G .

We need only to show that there is a homomorphism of Ψ onto Γ ; that its kernel is Ω follows immediately from the definition of Ω .

Let ψ_1, ψ_2 be members of Ψ , representing automorphisms γ_1, γ_2 of G . Then we show that $\psi_1\psi_2$ represents $\gamma_1\gamma_2$.

If $\gamma_2(a_i) = a_j$ and $\gamma_1(a_j) = a_k$, then $\gamma_1\gamma_2(a_i) = a_k$. From the relationship between ψ_1 and γ_1 , and ψ_2 and γ_2 , $\psi_2(a_i) = a_j$ and $\psi_1(a_j) = a_k$, so $\psi_1\psi_2(a_i) = a_k$. And also if in addition $\gamma_2(a_i') = a_j'$ and $\gamma_1(a_j') = a_k'$, $(\gamma_1\gamma_2(a_i), \gamma_1\gamma_2(a_i')) = (a_k, a_k')$ and $\psi_2\psi_1$ maps the line segment (a_i, a_i') onto (a_k, a_k') .

Theorem 4.5.3.

Ψ and Γ are isomorphic if and only if Ω is the identity function only.

This is an immediate deduction.

Finally in this general discussion of the properties of symmetric embeddings, suppose we have two groups Φ_1 and Φ_2 of functions in R^n .

Theorem 4.5.4.

Let Φ_1 and Φ_2 be two groups of one-to-one functions of R^n onto itself, with Φ_2 a subgroup of Φ_1 . Then if G' is any Φ_2 -symmetric embedding of a graph G into R^n , G' is also a Φ_1 -symmetric embedding of G into R^n .

Again the proof is immediate.

4.6 We now turn to consider several groups Φ .

The widest possible class consists of all one-to-one functions of R^n onto itself. In this case we have the simple result that a graph can be symmetrically embedded in R^n if and only if it can be embedded in R^n . For let γ be an automorphism of G with $\gamma(a_i) = a_j$ and $\gamma(a_i') = a_j'$. We define the following function on R_n :

If \underline{x} is a vertex \underline{a}_i of G' , $\phi(\underline{a}_i) = \underline{a}_j$.

If \underline{x} is a point $r\underline{a}_i + (1-r)\underline{a}_i'$ with $0 \leq r \leq 1$,
 $\phi(\underline{x}) = r\underline{a}_j + (1-r)\underline{a}_j'$. (These are the points on the line segments. As no point other than the appropriate vertices is on more than one line, the function is properly defined.)

If \underline{x} is not in G' , $\phi(\underline{x}) = \underline{x}$.

As a corollary, every graph can be symmetrically embedded in R^3 with this group.

4.7 A much more useful group is the set of all homeomorphisms of R^n onto itself. An embedding which is symmetric under this group is given the special name of a homeomorphic symmetric embedding.

For homeomorphic symmetric embeddings Ψ (of section 4.5) is never isomorphic to Γ , the group of G , for it is always possible to construct a non-identity homeomorphism of R^n onto itself which induces the identity in G' and so in G . One such is the following:

Let p be any point in R_n but not in G' . Then there is a spherical neighbourhood $N(p,a)$ centred at p and radius a which does not meet G' . Let q be any point in this neighbourhood, distant $|q-p|$ from p . Then the transformation which maps q to $p + \frac{|q-p|}{a}(q-p)$, and leaves every point outside $N(p,a)$ fixed.

It may of course still be possible to find a group Θ of homeomorphisms inducing Γ in G which is isomorphic to Γ .

We next consider which graphs can be homeomorphically symmetrically embedded in the various real coordinate spaces. We shall be most concerned with the possibility of embedding the complete graph K_n in the space R^n .

4.8 Very few graphs can be homeomorphic symmetrically embedded in R^1 . Any embedding in R^1 induces an ordering in the vertices which must be either held fixed or reversed completely by every homeomorphism of the line into itself. Thus for a finite graph the only possible groups are the trivial one and the group of order 2. Among finite graphs there are only the graph consisting of a single isolated vertex, the complete and null graphs K_2 and N_2 on two vertices, and the graphs with vertices a_1, \dots, a_n and edges (a_r, a_{r+1}) for all $r = 1, \dots, n-1$, for each positive integer n . Among the infinite graphs with finite groups there is the graph with one component of type I (see section 4.2) and with two such components, and also the graphs formed from each of these by adding an isolated vertex. Finally the infinite graph of type II can be homeomorphic symmetrically embedded and has an infinite group.

4.9 By no means every planar graph can be homeomorphic symmetrically embedded in R^2 . Considering first the complete graphs, we can state

Theorem 4.9.1.

The complete graph K_3 on 3 vertices can be homeomorphic symmetrically embedded in R^2 .

The construction of such an embedding is trivial: see section 5.3 for an explicit embedding.

On the other hand, although K_4 can be embedded in \mathbb{R}^2 ,

Theorem 4.9.2.

The complete graph K_4 on 4 vertices cannot be homeomorphic symmetrically embedded in \mathbb{R}^2 .

If four points are taken in a plane, then either they form the vertices of a convex quadrilateral, or one lies within the triangle formed by the other three, or three or more of them are collinear. In this last case the four points cannot be taken as the vertices of an embedding of K_4 in the plane. In the first case it may be shown (e.g. Mo 2 p 71) that the diagonals of the convex quadrilateral so defined intersect, so there is again no embedding. Hence only the second case remains.

But this case effectively differentiates the vertex within the triangle of the other three from the others. For the edges of the embedding divide the plane into one unbounded region and three bounded ones (fig. 4.9.1). Then a homeomorphism generating automorphisms on the vertices also permutes these regions, including mapping a bounded region onto the unbounded region. But while the former is compact, the latter is not, yielding a contradiction. Hence there is no homeomorphic symmetric embedding of K_4 .

Theorem 4.9.3.

The null graph N_n on n vertices can be homeomorphically

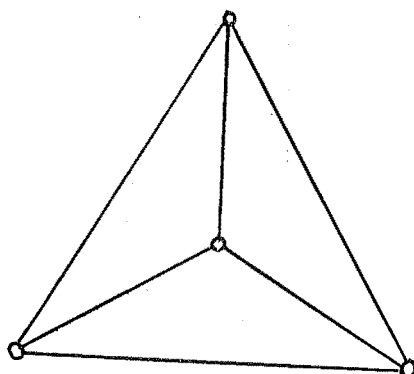


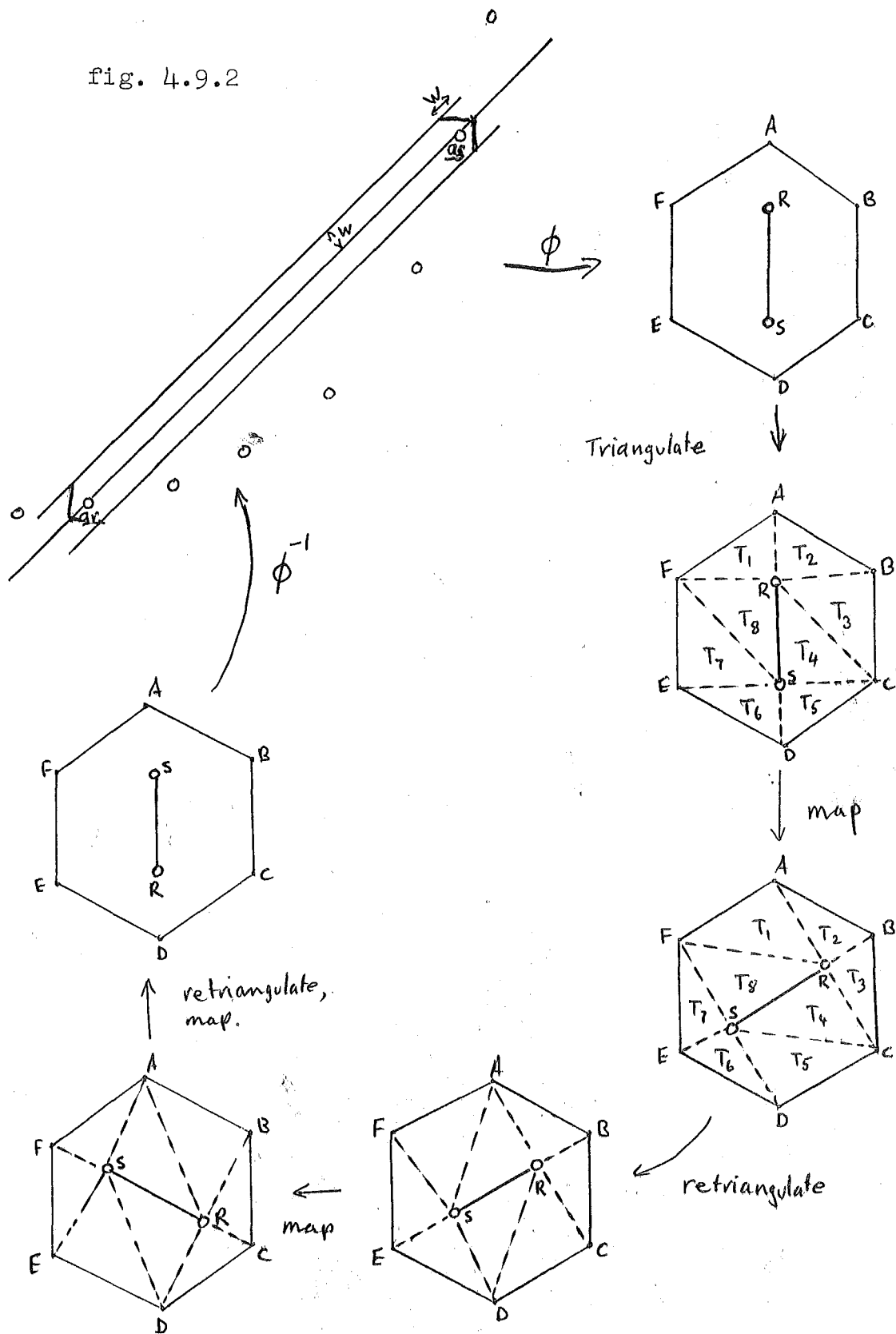
fig. 4.9.1.

symmetrically embedded in \mathbb{R}^2 .

First we remark that every permutation of the vertices is an automorphism, and then that every permutation can be broken down into a product of transpositions. We have therefore only to provide a homeomorphism which will interchange two vertices of G' and leave the rest fixed.

We take the vertices of N_n as a_1, \dots, a_n , and let $\underline{a}_r = (r, r^2)$. Then two vertices in the embedding may be joined by a straight line which does not pass through any other vertices. Indeed an infinite strip may be found of which the line through \underline{a}_r and \underline{a}_s is the axis, containing no other vertices. If the width of this strip is $2w$, points may be found on the axis at distances w beyond \underline{a}_r and \underline{a}_s . We also find the four points which are the feet of the perpendiculars from \underline{a}_r and \underline{a}_s onto the edges of the strip. These six points can then be joined up to form a hexagon containing \underline{a}_r and \underline{a}_s , as in fig. 4.9.2. This hexagon is

fig. 4.9.2



homeomorphic to a regular hexagon in which the line segment $(\underline{a}_r, \underline{a}_s)$ is mapped into a line segment (R, S) . Let this homeomorphism be ϕ .

The next stage is to define a homeomorphism of a regular hexagon onto itself which rotates a line segment of its interior through 60 degrees. This is again illustrated in the figure. We divide the hexagon into eight triangles T_1 to T_8 and map each independently by an affine transformation as shown. (Given any two triangles there is always an affine mapping of one onto the other.) As the affine transformations agree over the boundaries of the triangles, the mapping over the whole hexagon is properly defined and a homeomorphism. Moreover the mapping keeps the boundary of the hexagon fixed. (See section 6.5 on affine transformations.)

The hexagon is then re-triangulated in the original manner (but with the axis rotated through 60 degrees) and the internal line segment rotated again. This whole procedure is repeated a third time, so that the internal line segment has now been completely reversed, but the perimeter of the hexagon has been held fixed. The points R and S have been interchanged. The inverse of ϕ maps the hexagon back into R^2 , and we find that \underline{a}_r and \underline{a}_s have been interchanged, and the boundary left fixed. The composition of all these mappings is a homeomorphism of R^2 into itself which interchanges \underline{a}_r and \underline{a}_s but keeps the other vertices fixed.

The method also applies of course to the null graph

with a countable number of vertices.

4.10 The transformation used in the previous section to rotate the hexagon is in fact a piecewise affine transformation.

Definition 4.10.1.

Let R^n be covered by a locally finite collection $\{C_i\}$ of closed sets, each mapped by an affine transformation ϕ_i into R^m such that

- (1) $\{\phi_i(C_i)\}$ is locally finite
- (2) $\bigcup_i (\phi_i(C_i)) = R^m$
- (3) If $\underline{x} \in C_i \cap C_j$, $\phi_i(\underline{x}) = \phi_j(\underline{x})$.

Then the mapping ϕ defined by

$$\phi(\underline{x}) = \phi_i(\underline{x}) \text{ where } C_i \text{ is any member of the collection containing } \underline{x}.$$

is called a piecewise affine transformation of R^n onto R^m .

Since each ϕ_i is continuous, ϕ is also continuous (Hu 2 p 33). If in addition ϕ is one-to-one and onto (when necessarily $m = n$), ϕ is termed invertible. Each ϕ_i is then also invertible. The invertible piecewise affine transformations are at once homeomorphisms. They may also be shown to form a group.

As the invertible piecewise affine transformations are simple to define and also have other useful properties we will use them to provide homeomorphic symmetric

embeddings. Theorem 4.5.4 shows that every piecewise affine symmetric embedding is also a homeomorphic symmetric embedding. In fact I conjecture that the two classes of graph symmetrically embeddable according to the two groups coincide.

The regions C_i may be taken to be (possibly infinite) n -dimensional polyhedra with the intersection of any two empty or a face of each. For if there is a set of $n+1$ independent points in $C_i \cap C_j$ it can be shown that the affine transformations defined on them are identical.

4.11 In the development of this thesis the linear symmetric embeddings were studied first and only later was the concept widened to other types of symmetric embedding. Hence I here simply state the result of Theorem 5.3.1:

"The complete graph K_n on n vertices can be linear symmetrically embedded in R^{n-1} ."

Therefore K_n can be homeomorphic symmetrically embedded in R^{n-1} . Now it is shown in section 5.7 that K_n cannot be linear symmetrically embedded in R^{n-2} . It is therefore an interesting problem, which can only be very incompletely answered here, to find the relationship between m and n if K_n can be homeomorphic symmetrically embedded in R^m but not in R^{m-1} .

Theorem 4.11.1.

K_5 can be piecewise-affine and therefore homeomorphic

symmetrically embedded in E^3 .

We prove this by defining an embedding and a set of transformations corresponding to the generators of K_5 .

Let the five vertices of G' be

$$\begin{aligned} \underline{u}_0 &= (0,0,0) & \underline{u}_1 &= (-e_1, -e_2, -e_3) & \underline{u}_2 &= (e_1, -e_2, -e_3), \\ \underline{u}_3 &= (0, 2e_2, -e_3) & \underline{u}_4 &= (0, 0, 3e_3) \end{aligned}$$

$$\text{where in fact } e_1 = \sqrt{\frac{2}{3}}, \quad e_2 = \frac{\sqrt{2}}{3}, \quad e_3 = \frac{1}{3}.$$

(\underline{u}_1 to \underline{u}_4 are the embedding of K_4 in R^3 defined in theorem 5.3.1).

These points are then the centroid and vertices of a regular tetrahedron, the vertices being at a distance 1 from the centroid.

Each automorphism of K_5 can be broken down into a product of transpositions, each of which is itself an automorphism of K_5 . It therefore suffices to construct a homeomorphism for each transposition.

The transpositions are of two kinds: those involving only $\underline{u}_1, \underline{u}_2, \underline{u}_3$ and \underline{u}_4 , and those involving \underline{u}_0 . The first kind are easily dealt with by a reflection in the plane formed by the non-involved vertices. For instance the transposition interchanging \underline{u}_1 and \underline{u}_2 is induced by a reflection in the plane $x_1 = 0$, which contains $\underline{u}_0, \underline{u}_3$ and \underline{u}_4 . The point (x_1, x_2, x_3) is mapped to $(-x_1, x_2, x_3)$.

The second kind can effectively be reduced to the

single case of the interchange of u_0 and u_4 , since any other transposition involving u_0 can be expressed as a product of this transposition with two others of the first kind. It remains therefore to construct an invertible piecewise affine transformation to interchange u_0 and u_4 , leaving the other vertices fixed.

We divide R^3 into four regions by planes, each parallel to the x_3 -axis. The first region R_0 consists of all those points whose projection onto the plane $x_3 = 0$ lies on or outside the boundary of the triangle $\underline{u}_1, \underline{u}_2, \underline{u}_3$. This is the union of the regions $x_2 \leq -e_2$, $3e_2x_1 + e_1x_2 \geq 2e_1e_2$ and $3e_2x_1 - e_1x_2 \leq -2e_1e_2$.

The complement of R_0 is then a triangular prism whose cross-section is the triangle $\underline{u}_1, \underline{u}_2, \underline{u}_3$ and whose axis is the x_3 -axis. This is divided by the planes P_1 through $\underline{u}_0, \underline{u}_1$ and \underline{u}_4 , P_2 through $\underline{u}_0, \underline{u}_2$ and \underline{u}_4 and P_3 through $\underline{u}_0, \underline{u}_3$ and \underline{u}_4 into the regions R_1, R_2, R_3 :

$$R_1: 3e_2x_1 + e_1x_2 \leq 2e_1e_2, x_1 \geq 0, x_1e_2 + e_1x_2 \geq 0.$$

$$R_2: 3e_2x_1 - e_1x_2 \geq -2e_1e_2, x_1 \leq 0, x_1e_2 - x_2e_1 \leq 0.$$

$$R_3: x_2 \geq -e_2, e_2x_1 + e_1x_2 \leq 0, x_1e_2 - x_2e_1 \geq 0.$$

The transformation is divided into two steps. In the first step R_0 is held fixed and affine transformations applied in the other regions to keep $\underline{u}_1, \underline{u}_2$ and \underline{u}_3 fixed and to send \underline{u}_0 to $(0, 0, -5e_3)$ and \underline{u}_4 to $(0, 0, -2e_3)$.

The transformations are:

$$R_0: \phi_0: x_1' = x_1, x_2' = x_2, x_3' = x_3$$

$$R_1: \phi_1: x_1' = x_1, x_2' = x_2, x_3' = x_3 + \frac{5e_3}{2e_1e_2}(3e_2x_1 + e_1x_2) - 5e_3$$

$$R_2: \phi_2: x_1' = x_1, x_2' = x_2, x_3' = x_3 - \frac{5e_3}{2e_1e_2}(3e_2x_1 - e_1x_2) + 5e_3$$

$$R_3: \phi_3: x_1' = x_1, x_2' = x_2, x_3' = x_3 - \frac{5e_3}{e_2}x_2 + 5e_3$$

These mappings are identical over the intersection of any two regions. The second stage consists of reflecting the whole space in the plane $x_3 = -e_3$:

$$x_1' = x_1; \quad x_2' = x_2, \quad x_3' = -2e_3 - x_3.$$

Hence \underline{u}_4 is mapped to $(0,0,0)$ and \underline{u}_0 to $(0,0,3e_3)$.

The other vertices are held fixed throughout as are each of the lines $(\underline{u}_1, \underline{u}_2), (\underline{u}_2, \underline{u}_3), (\underline{u}_1, \underline{u}_3)$. The lines $(\underline{u}_0, \underline{u}_1)$ and $(\underline{u}_4, \underline{u}_1)$ are interchanged and so are $(\underline{u}_0, \underline{u}_2)$ and $(\underline{u}_4, \underline{u}_2)$ and $(\underline{u}_0, \underline{u}_3)$ and $(\underline{u}_4, \underline{u}_3)$, while the line $(\underline{u}_0, \underline{u}_4)$ is reversed.

4.12 Our next aim is to show that this is the best that can be attained: K_6 cannot be homeomorphic symmetrically embedded in R^3 . Our proof will be lengthy, covering two sections, the first of which will establish important concepts and a vital theorem. We here lean on certain separation properties of the plane and three-dimensional space. These are only stated here but may be established by continuing an approach such as Moise (Mo 2) chapter 4.

Theorem 4.12.1.

Let l_1, l_2 be two lines intersecting in a point p .
 Let a_1, b_1 be points on l_1 , distinct from each other and p in the order p, a_1, b_1 and a_2, b_2 points on l_2 , distinct from each other and p in the order p, a_2, b_2 . Then $a_1 b_1 b_2 a_2$ is a convex quadrilateral. Hence, from (Mo 2 p 71 th. 1), referred to in Theorem 4.9.2, $a_1 b_2$ and $a_2 b_1$ intersect.

Theorem 4.12.2.

Let p, q, r, s be four points in space, not in the same plane. Let x be a point within the triangle $p r s$ (i.e. a member of the interior of $p r s$ in the plane defined by that triangle), and y a point on $p q$, between those points. Then the line segment $x q$ meets the plane of the triangle $r s y$ in a point z within $r s y$.

Theorem 4.12.3.

With p, q, r, s and y defined as in the previous theorem, let x instead be a point in the plane of $p r s$ but outside the triangle. Then $x q$ meets the plane of $r s y$ in a point z outside $r s y$.

Theorem 4.12.4.

Let $a b c$ be a triangle, and let p be a point within $a b c$. Then the union of the triangles $p a b$, $p b c$, $p c a$ together with their interiors is $a b c$ together with its interior.

Besides these theorems we have occasionally assumed other more direct consequences of the separation properties.

Consider two triads of points $\underline{a}_1, \underline{b}_1, \underline{c}_1$ and $\underline{a}_2, \underline{b}_2, \underline{c}_2$ in R^3 . Each determines a triangle T_i , consisting of these points and the line segments between them, and also a plane P_i , in which the three points lie. If the planes are neither coincident nor parallel they meet in a line L . Again, this line L may or may not meet either triangle. If it does, it does so either in a single vertex, or in two points on different edges or in two vertices and the whole edge between them, or in a vertex and a point of the opposite edge.

We assume throughout this discussion that the sets T_1 and T_2 as defined above have no points in common. Then if the line L is defined and meets both triangles there are three cases which we need to distinguish.

(a) $T_1 \cap L$ and $T_2 \cap L$ each consist of two points, and there is a point of $T_1 \cap L$ between the points of $T_2 \cap L$, and a point of $T_2 \cap L$ between the points of $T_1 \cap L$. In this case the triangles are said to be linked. In all other cases when L is defined, and in all cases when L is not defined, T_1 and T_2 are said to be unlinked.

(b) At least one of $T_1 \cap L$ and $T_2 \cap L$ consists of two points. If $T_1 \cap L$ consists of two points and there is at least one point of $T_2 \cap L$ between these points, but not a point of $T_1 \cap L$ between the points of $T_2 \cap L$ (if

there is more than one member of this set), then T_j is said to penetrate T_i . We also say that T_j penetrates T_i if the planes P_i and P_j coincide and T_j lies within T_i (i.e. any line L' which cuts T_j has the same properties as L above).

(c) There is a point of L which lies between all points of $T_1 \cap L$ and $T_2 \cap L$.

These three cases are shown in fig. 4.12.1.

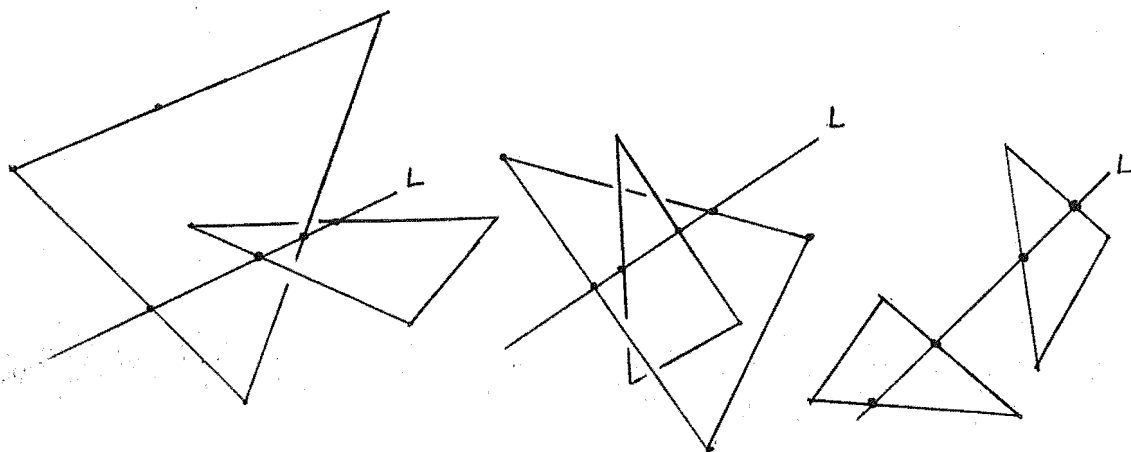


fig. 4.12.1.

It is well worth proving the following theorem as a criterion for linked triangles:

Theorem 4.12.5.

Let T_1 be a triangle with vertices $\underline{a_1b_1c_1}$, and T_2 a triangle with vertices $\underline{a_2b_2c_2}$. Then if T_2 meets the plane

of T_1 in two points, one of which is within T_1 and the other outside T_1 , T_1 and T_2 are linked.

Let \underline{p} and \underline{q} be these two points. Then L , the intersection of the planes of T_1 and T_2 passes through \underline{p} and \underline{q} . As \underline{p} is inside and \underline{q} outside, there is a point \underline{x} of T_1 on the line between them. Also as \underline{p} is inside T_1 there are points of T_1 on both sides of it, hence another point \underline{y} , on the side of \underline{p} away from \underline{x} . Then \underline{p} lies between the two points of $T_1 \cap L$ and \underline{x} between the points of $T_2 \cap L$.

Theorem 4.12.6.

Let $\underline{a}_1, \underline{b}_1, \underline{c}_1, \underline{a}_2, \underline{b}_2, \underline{c}_2$ be six points in R^3 , so placed that no two edges determined by any two disjoint pairs of these points intersect, and such that the triangle T_1 determined by $\underline{a}_1, \underline{b}_1, \underline{c}_1$ penetrates the triangle T_2 determined by $\underline{a}_2, \underline{b}_2, \underline{c}_2$. Then there is at least one pair of linked triangles determined by the six vertices.

The conditions do not prevent four of the points being coplanar, provided that one of the points lies within the triangle formed by the other three (section 4.9), but they do prevent five points from being coplanar (Theorem 4.3.1).

Let P_2 be the plane of T_2 , and consider the possible dispositions of three other points of R^3 . The cases are

- (i) Three in the plane P_2
- (ii) Two in P_2 , one out of it
- (iii) One in P_2 , two on the same side of P_2

- (iv) One in P_2 , the others on different sides of P_2
- (v) Two on one side of P_2 , one on the other
- (vi) Three on the same side of P_2 .

If now these three points are the vertices of T_1 , in accord with the hypothesis of the theorem, (i) and (ii) can be dismissed at once, since they give respectively six and five points in the plane. Case (vi) can also be dismissed for then T_1 does not penetrate T_2 .

Case (iii) Let \underline{c}_1 be the vertex of T_1 in P_2 . Then we distinguish two cases.

(a) the infinite line $\underline{a}_1\underline{b}_1$ meets P_2 in a point \underline{p} within or on T_2 . ('on T_2 ' means that \underline{p} is a vertex of T_2 or belongs to one of its edges.) Let \underline{a}_1 lie between \underline{p} and \underline{b}_1 .

(b) either $\underline{a}_1\underline{b}_1$ is parallel to P_2 or the point \underline{p} defined above is outside T_2 .

In either case we write $\underline{c}_1 = \underline{q}$ and refer to the case (w) below for (a) and (x) for (b).

Case (iv) Let \underline{c}_1 be the vertex of T_1 in P_2 , and let the infinite line $\underline{a}_1\underline{b}_1$ cut P_2 in \underline{p} . Then \underline{p} lies between \underline{a}_1 and \underline{b}_1 , and \underline{c}_1 and \underline{p} lie within T_2 as T_1 penetrates T_2 .

As \underline{c}_1 lies within T_2 , we express T_2 and its interior as the union of the triangles $\underline{c}_1\underline{a}_2\underline{b}_2$, $\underline{c}_1\underline{b}_2\underline{c}_2$, $\underline{c}_1\underline{c}_2\underline{a}_1$ and their interiors. As $\underline{a}_1\underline{b}_1$ must not meet $\underline{c}_1\underline{a}_2$, $\underline{c}_1\underline{b}_2$ or $\underline{c}_1\underline{c}_2$, \underline{p} lies in the interior of (at least) one of these. Suppose, without loss of generality that \underline{p} lies within $\underline{c}_1\underline{a}_2\underline{b}_2$. Then

we will show that the triangles $\underline{c_1a_2b_2}$ and $\underline{c_2a_1b_1}$ are linked.

For the plane of $\underline{c_1a_2b_2}$ is P_2 . We consider the points of intersection of $\underline{c_2a_1b_1}$ with P_2 . These are at once \underline{p} and $\underline{c_2}$. Now \underline{p} is within $\underline{c_1a_2b_2}$ and since $\underline{c_1}$ is within $\underline{a_2b_2c_2}$, $\underline{c_2}$ is not within $\underline{c_1a_2b_2}$. Hence applying theorem 4.12.5 the triangles are linked.

This case is shown in fig. 4.12.2.

Case (v) Again there are two cases, similar to those in case (iii). We let $\underline{a_1}, \underline{b_1}$ be the two points on the same side of P_2 . If the infinite line through $\underline{a_1}$ and $\underline{b_1}$ meets P_2 in a point \underline{p} within or on T_2 , let $\underline{a_1}$ lie between $\underline{b_1}$ and \underline{p} , and let the intersection of $\underline{b_1c_1}$ with P_2 be \underline{q} . We then refer to case (w). If on the other hand the point \underline{p} is not defined or lies outside T_2 we refer to case (x), again letting $\underline{b_1c_1}$ meet P_2 in \underline{q} . Since T_1 penetrates T_2 , \underline{q} lies within T_2 in each case.

Case (w) The points \underline{p} and \underline{q} lie in the planes of both T_1 and T_2 , so in the line L . We have assumed that \underline{p} lies within or on T_2 ; \underline{q} lies within T_2 since T_1 penetrates T_2 . Now \underline{pq} produced beyond \underline{q} must meet T_2 , but it does not meet it in a vertex. For suppose for example that \underline{pq} produced passed through $\underline{c_2}$. Then by Theorem 4.12.1 $\underline{a_1c_2}$ would meet $\underline{b_1q}$. And whether \underline{q} equals $\underline{c_1}$ or not, $\underline{b_1q}$ is a segment of $\underline{b_1c_1}$, so $\underline{a_1c_2}$ would meet $\underline{b_1c_1}$, contrary to hypothesis. Let the triangle T_2 be so labelled that \underline{pq} produced meets T_2 in a point \underline{s} on $\underline{a_2c_2}$. Then \underline{q} is a point within the triangle

$\underline{a_2 p c_2}$. On the other hand $\underline{b_2}$ is not within $\underline{a_2 p c_2}$. Applying theorems 4.12.2 and 4.12.3 to the triangles $\underline{a_2 p c_2}$ and $\underline{a_2 a_1 c_2}$ and the segment $\underline{b_1 c_1}$ we find that $\underline{b_1 c_1}$ meets the plane of $\underline{a_2 a_1 c_2}$ within that triangle, while $\underline{b_1 b_2}$ meets that plane outside the triangle. Applying theorem 4.12.5 we find that the triangles $\underline{a_2 a_1 c_2}$ and $\underline{b_1 b_2 c_1}$ are linked. This case, with $\underline{c_1}$ not in the plane, is shown in fig. 4.12.3.

Case (x) Let L be the line of intersection of the planes of T_1 and T_2 . Then L contains \underline{g} , whichever previous case has given rise to this case and meets T_2 in two points \underline{r} and \underline{s} .

Now neither of $\underline{r}, \underline{s}$ is a vertex of T_2 . For suppose \underline{s} is $\underline{a_2}$. Then $\underline{a_1}, \underline{b_1}, \underline{c_1}$ and $\underline{a_2}$ all lie in one plane, and no one lies within the triangle formed by the other three, for varying reasons. If $\underline{a_2}$ lies within $\underline{a_1 b_1 c_1}$, T_1 does not penetrate T_2 . If $\underline{a_1}$ or $\underline{b_1}$ lies in the triangle of the other three, the infinite line $\underline{a_1 b_1}$ meets P_2 within T_2 . If $\underline{c_1}$ lies within $\underline{a_1 b_1 a_2}$, $\underline{c_1}$ lies on the same side of P_2 as $\underline{a_1}$ and $\underline{b_1}$. Hence the assumption that \underline{s} is $\underline{a_2}$ leads to a contradiction. Similarly in the other cases. We fix the labelling of the diagram so that \underline{r} lies on $\underline{b_2 c_2}$ and \underline{s} on $\underline{a_2 c_2}$; also \underline{r} and \underline{s} are chosen, relative to $\underline{a_1}$ and $\underline{b_1}$ so that $\underline{r s b_1 a_1}$ in that order is a convex quadrilateral. (Either $\underline{r s}$ and $\underline{a_1 b_1}$ are parallel or they intersect if produced at a point \underline{u} . Then we take $\underline{a_1}$ between $\underline{b_1}$ and \underline{u} and \underline{r} between \underline{s} and \underline{u} .)

Consider the triangles $\underline{a_1 a_1 c_2}$ and $\underline{b_1 b_2 c_2}$. Their planes are not parallel, for they have a point in common.

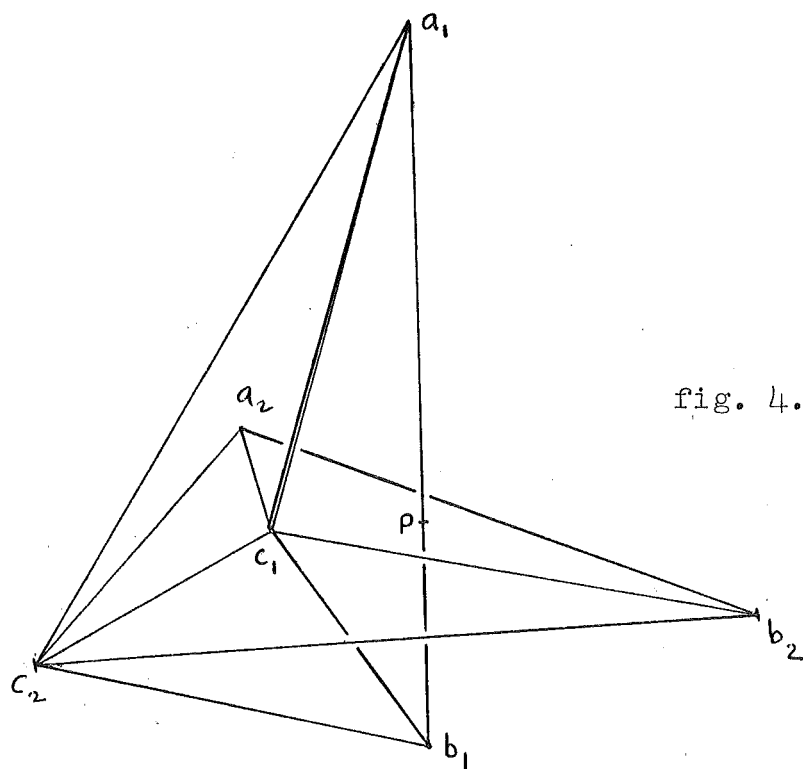


fig. 4.12.2

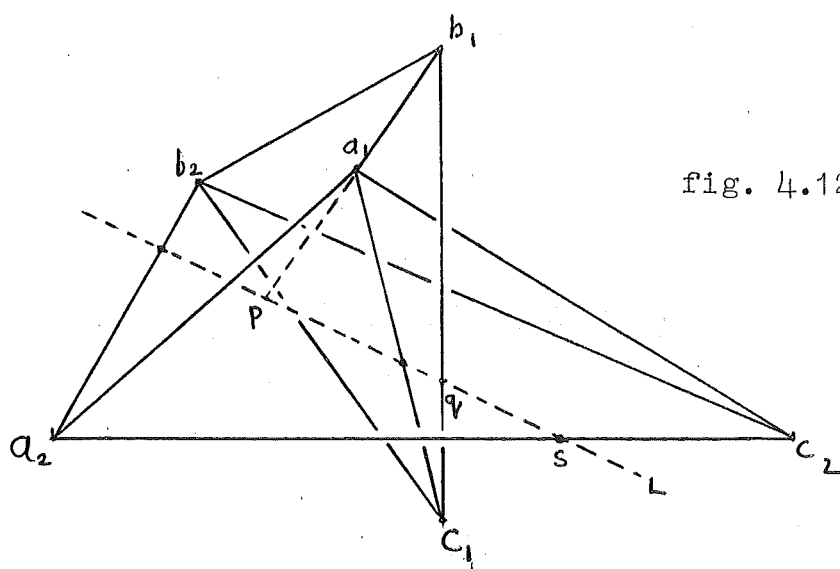


fig. 4.12.3

Therefore they meet in a line. Let this line be L' . Take first the plane defined by $\underline{a_1a_2c_2}$. On one side of this plane lies $\underline{b_2}$, and all points within T_2 . If $\underline{b_1}$ lies on this side of the plane, so does the whole of the line $\underline{a_1b_1}$ and its continuation beyond $\underline{b_1}$. But that would make $\underline{a_1b_1}$ meet the plane P_2 within T_2 , since $\underline{a_1b_1}$ also lies in the plane of T_1 , making the intersection with P_2 lie on \underline{rs} . Hence $\underline{b_1}$ and $\underline{b_2}$ are on opposite sides of the plane of $\underline{a_1a_2c_2}$, so the line L' meets the line segment $\underline{b_1b_2}$ in a point $\underline{b_3}$, say. Similarly L' meets $\underline{a_1a_2}$ in a point, which we will call $\underline{a_3}$. Now $\underline{a_3}, \underline{b_3}$ do not coincide, or else $\underline{b_1b_2}$ intersects $\underline{a_1a_2}$.

We need to distinguish two cases according as $\underline{a_3}$ lies between $\underline{b_3}$ and $\underline{c_2}$ or $\underline{b_3}$ lies between $\underline{a_3}$ and $\underline{c_2}$. The two cases are essentially obtained from one another by relabelling, and we shall consider only the former case, which is shown in fig. 4.12.4. We will then show that $\underline{b_1b_2c_1}$ and $\underline{a_1a_2c_2}$ are linked.

We already know that $\underline{b_1b_2}$ cuts the plane of $\underline{a_1a_2c_2}$ in $\underline{b_3}$, which as it lies on L' beyond $\underline{a_3}$, is outside $\underline{a_1a_2c_2}$. It remains only to show that $\underline{b_1c_1}$ cuts the plane of $\underline{a_1a_2c_2}$ within that triangle, and then to invoke Theorem 4.12.5. We know that $\underline{b_1c_1}$ cuts P_2 in \underline{g} , which lies within T_2 : \underline{g} may in fact be $\underline{c_1}$. We have already determined that $\underline{rsb_1a_1}$ is a convex quadrilateral, so the diagonals $\underline{a_1s}, \underline{b_1r}$ meet. But \underline{g} lies between \underline{r} and \underline{s} , so $\underline{b_1g}$ also meets $\underline{a_1s}$ in a point we will call \underline{t} . Then \underline{t} lies on $\underline{b_1c_1}$ and within

fig. 4.12.4

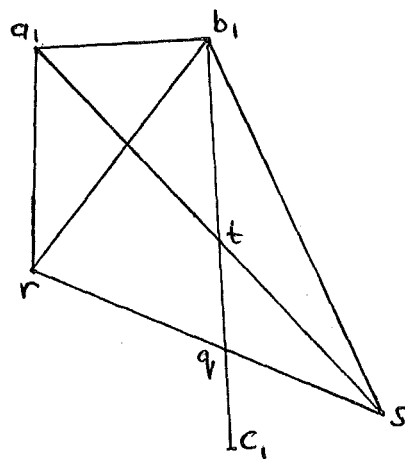
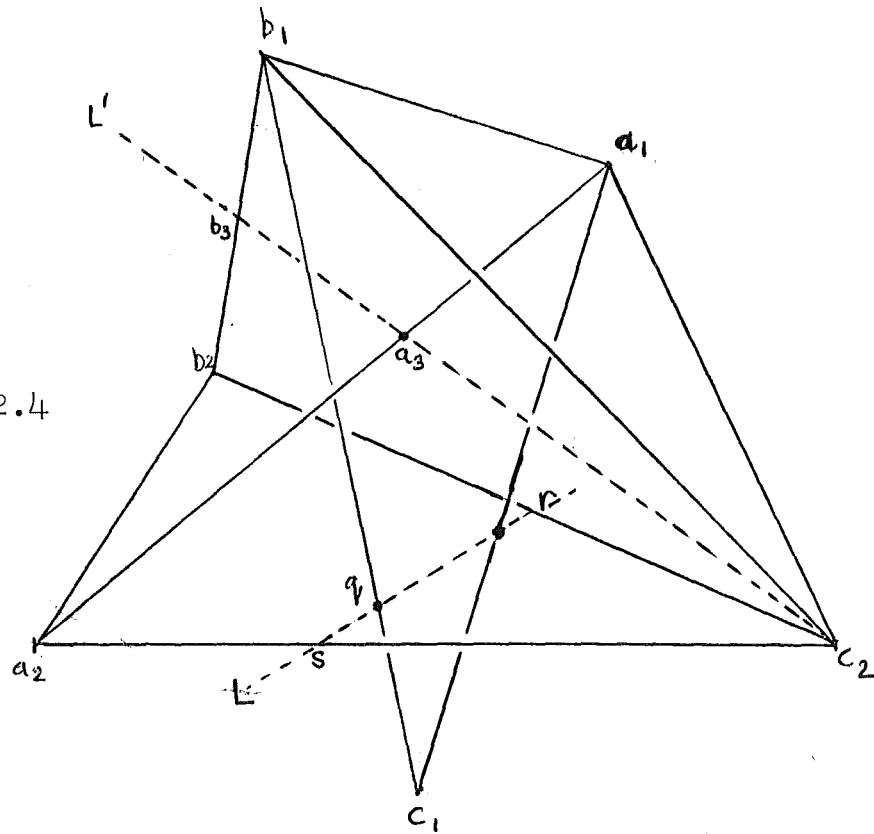


fig. 4.12.5

$a_1 a_2 c_2$. (See fig. 4.12.5). This establishes our final case and hence the theorem.

4.13 Having proved theorem 4.12.6 the rest is relatively easy, but we shall need this time several results and concepts of topology. We first state two theorems:

Theorem 4.13.1.

There is no homeomorphism of R^3 onto itself which maps a linked pair of triangles onto an unlinked pair.

The triangles are all homeomorphic to the circle. The result is proved by showing that the homotopy groups of the two complement spaces (omitting the points on the triangles) are not isomorphic.

Theorem 4.13.2.

The sphere with one cross-cap (real projective plane) cannot be embedded in R^3 .

For the topology of surfaces see for instance (ST 2). The crucial theorem is on page 222.

We are now ready to prove our main theorem.

Theorem 4.13.3.

There is no homeomorphic symmetric embedding of the complete graph K_6 in R^3 .

Suppose we have such an embedding. There are ten

different ways of dividing the vertices of K_6 into two sets of three, and each defines a pair of triangles with no points in common. Each automorphism of K_6 generates a homeomorphism of R^3 which maps each pair of triangles onto some pair of triangles, and given any two pairs there is a homeomorphism of R^3 mapping one pair onto the other. Thus in view of Theorem 4.13.1 either every pair is linked or every pair is unlinked.

But there is certainly a plane which divides R^3 into two regions, each containing three vertices of K_6 , and the corresponding pair of triangles is certainly unlinked. Hence there must be no linked pairs of triangles, and in virtue of Theorem 4.12.6 there can be no penetrating pairs either.

Our aim is now to show that if $\underline{a_1b_1c_1}, \underline{a_2b_2c_2}$ are two triads of vertices of K_6 , not necessarily disjoint, then the 2-simplexes (triangles and interiors) either do not meet at all, or in an edge of each, or in a single vertex, so that unless the triads coincide, the 2-simplexes have no interior points in common. We have already shown that if the triads are disjoint the 2-simplexes are also disjoint if we have a homeomorphic symmetric embedding.

Now suppose that the two triads have one vertex in common. Let this be $\underline{a_1}$, and the triads $\underline{a_1a_2a_3}, \underline{a_1a_4a_5}$. All five vertices cannot be in the same plane, since K_5 cannot be embedded in the plane. Let the 2-simplexes have a point \underline{x} in common. Then if \underline{x} lies on edges of both triangles,

the embedding is improper, since two different edges have a point in common. Suppose it lies within at least $\underline{a_1 a_2 a_3}$. Then \underline{x} is a point of the intersection L of the planes of the two triads, and $\underline{a_1}$ is another. Hence L also meets $\underline{a_2 a_3}$ in a point \underline{p} and $\underline{a_4 a_5}$ in a point \underline{q} (one of these, but not both, may be a vertex). The order of the points along L is either $\underline{a_1 x p q}$ or $\underline{a_1 x q p}$. We will assume the latter, and that \underline{q} is not $\underline{a_5}$ (leaving the possibility that \underline{q} is $\underline{a_4}$). We have several cases to consider.

First suppose that the remaining vertex $\underline{a_6}$ lies in the plane of $\underline{a_1 a_2 a_3}$. Then we can show at once that if $\underline{a_6}$ lies within $\underline{a_1 a_2 a_3}$, $\underline{a_4 a_5 a_6}$ penetrates $\underline{a_1 a_2 a_3}$, implying by Theorem 4.12.6 a linked pair of triangles somewhere. On the other hand if $\underline{a_6}$ is outside $\underline{a_1 a_2 a_3}$, that triangle is linked with $\underline{a_4 a_5 a_6}$.

On the other hand suppose $\underline{a_6}$ is not in the plane of $\underline{a_1 a_2 a_3}$. We know that $\underline{a_5}$ is not in that plane either. If $\underline{a_5}$ and $\underline{a_6}$ are on the opposite sides, let $\underline{a_5 a_6}$ cut the plane of $\underline{a_1 a_2 a_3}$ in \underline{r} . Then if \underline{r} lies outside $\underline{a_1 a_2 a_3}$, $\underline{a_1 a_2 a_3}$ and $\underline{a_4 a_5 a_6}$ are linked, and if \underline{r} lies inside $\underline{a_1 a_2 a_3}$, $\underline{a_4 a_5 a_6}$ penetrates $\underline{a_1 a_2 a_3}$, implying a linked pair somewhere.

If on the other hand $\underline{a_5}$ and $\underline{a_6}$ are on the same side, $\underline{a_4}$ and $\underline{a_6}$ are not (although $\underline{a_4}$ may be in the plane if it is also the point \underline{q}). Then we consider the point \underline{s} of intersection of $\underline{a_4 a_6}$ with the plane of $\underline{a_1 a_2 a_3}$. Again $\underline{a_4 a_5 a_6}$ either penetrates $\underline{a_1 a_2 a_3}$ or is linked with it.

Hence if two simplexes meet with one vertex in common, but also improperly, there is at least one linked pair of triangles.

We next need to consider the case when the two triads have two members in common. If two such 2-simplexes are to meet improperly they must be in the same plane. Let the triads be $\underline{a_1a_2a_3}$ and $\underline{a_1a_2a_4}$. Then also $\underline{a_3}$ and $\underline{a_4}$ must lie on the same side of $\underline{a_1a_2}$ (in their common plane), and unless one of $\underline{a_3}, \underline{a_4}$ lies within the triangle formed by the other vertices, there will be a pair of edges which meet. We suppose that $\underline{a_4}$ lies within $\underline{a_1a_2a_3}$.

There are two more vertices $\underline{a_5}, \underline{a_6}$ to consider. Neither lies in the plane, for five vertices may not be coplanar. If both lie on the same side of the plane $\underline{a_4a_5a_6}$ penetrates $\underline{a_1a_2a_3}$, if on opposite sides, let $\underline{a_5a_6}$ meet the plane in \underline{p} . Then if \underline{p} lies inside $\underline{a_1a_2a_3}$ that triangle is penetrated by $\underline{a_4a_5a_6}$; if \underline{p} lies outside $\underline{a_1a_2a_3}$, that triangle is linked with $\underline{a_4a_5a_6}$.

Hence in any homeomorphic symmetric embedding of K_6 (in which we have shown there must be no linked triangles) in R^3 the 2-simplexes defined by the various triads of vertices either meet in an edge of each, or a vertex of each or not at all. Hence the 2-dimensional complex defined by all triads of the six vertices is properly embedded in R^3 .

But let us select the following ten 2-simplexes:

$(a_1a_2a_6), (a_2a_4a_6), (a_1a_2a_3), (a_2a_3a_5), (a_1a_3a_4), (a_1a_4a_5),$
 $(a_1a_5a_6), (a_3a_5a_6), (a_2a_4a_5), (a_3a_4a_6).$ These simplexes meet properly and every edge in the graph belongs to exactly two of these faces. They therefore define a closed surface. We find that there are ten 2-simplexes, 15 edges and 6 vertices, so the Euler-Poincare characteristic is $10 - 15 + 6 = 1$. The surface is then homeomorphic to the real projective plane (sphere with one cross-cap), which yields the desired contradiction with Theorem 4.13.2.

(The Euler-Poincare characteristic is constant for all triangulations on any surface. It is defined as

$$(\text{no. of faces}) - (\text{no. of edges}) + (\text{no. of vertices}).$$

For the plane this number is 2. See for instance (HY 1) pp 241-2.)

There seems little prospect of generalising this result to show that K_n cannot be homeomorphic symmetrically embedded in R^{n-3} . Even if this is so a completely different approach would appear to be necessary.

4.14 We next generalise Theorem 4.11.1.

Theorem 4.14.1.

The complete graph K_n can be homeomorphic symmetrically embedded in R^{n-2} for all $n \geq 5$.

Let the vertices of K_n be a_0, a_1, \dots, a_{n-1} . We take a linear symmetric embedding of the subgraph defined by

a_0, \dots, a_{n-3} in an $(n-3)$ -dimensional subspace P of R^{n-2} . (Again see Theorem 5.3.1.) We define P_i as the hyperplane containing each of these points except a_i for each $i = 0, \dots, n-3$. The origin is at the centroid of the vectors a_0, a_1, \dots, a_{n-3} , and the perpendiculars from the origin to the hyperplanes P_i all have the same length d .

As P is of dimension $n-3$ there is a unique direction perpendicular to it. We let the unit vector in this direction be \underline{e} (in either sense along the line). We give a_{n-2} the vector ze , where z is so chosen that a_0, \dots, a_{n-2} are the vertices of a regular $(n-2)$ -simplex. Although we do not need it, this gives the value $\sqrt{\frac{n}{n-2}}$ for z . If we then give a_{n-1} the vector $\frac{z}{n} \underline{e}$, it lies at the centroid of a_0, \dots, a_{n-2} .

This embedding is proper since the embedding of the subgraph determined by a_0, \dots, a_{n-3} is proper, no four vertices being in the same plane and no two of them are in any plane through a_{n-2} and a_{n-1} . Nor are any three of them coplanar with either of a_{n-2} or a_{n-1} .

As we have remarked before, every automorphism of a complete graph can be broken down into a product of transpositions, interchanging two vertices and keeping the rest fixed. Moreover the group is generated by transpositions in which one member is common to all the transpositions: we choose the class of transpositions in which a_{n-2} and one other vertex, for each of the other vertices, are interchanged. There are two cases to be distinguished:

- (a) transpose a_{n-2} and a_i for $i = 0, 1, \dots, n-3$
- (b) transpose a_{n-2} and a_{n-1} .

The first type is easily dealt with: we reflect in the hyperplane of points equidistant from a_{n-2} and a_i . This hyperplane contains all the other vertices, which are therefore mapped onto themselves, as are all edges between them. The vertices a_{n-2} and a_i are interchanged, and so are the edges (a_{n-2}, a_j) and (a_i, a_j) where $j = 0, 1, \dots, i-1, i+1, \dots, n-3$. The edge (a_i, a_{n-2}) is mapped onto itself reversed.

For the remaining transposition, interchanging a_{n-2} and a_{n-1} , we follow a generalisation of the method of Theorem 4.11.1.

Each point in R^{n-2} can be given a unique representation of the form $\underline{p} + q\underline{e}$, where \underline{p} is a vector in the subspace P . For each point \underline{x} in P let m_i be the perpendicular distance from \underline{x} to P_i , $i = 0, \dots, n-3$, positive if \underline{x} lies on the same side as the origin, negative on the other side. Now define m as the minimum of these. We associate now with each point $\underline{p} + q\underline{e}$ of R^{n-2} the same value of m as $\underline{p} + 0\underline{e}$ in P .

The next stage is to define the mapping ϕ which induces the desired transposition.

For all points with $m \leq 0$ we define $\phi(\underline{p} + q\underline{e})$ to be the mirror image in P :

$$\phi(\underline{p} + q\underline{e}) = \underline{p} - q\underline{e}.$$

The maximum positive value of m is d , as the origin

as centroid of the symmetrically placed points $\underline{a}_0, \dots, \underline{a}_{n-3}$ is the maximum positive minimum distance from the hyperplanes P_i . For points with $m \geq 0$ we put

$$\phi(\underline{p} + q\underline{e}) = \underline{p} + \frac{mZ}{d}(1 + \frac{1}{n})\underline{e} - q\underline{e}.$$

When $m = 0$ this reduces to $\underline{p} - q\underline{e}$.

We have thus defined ϕ on two closed regions, whose union is R^{n-2} , with the definitions agreeing on the intersection, the set with $m = 0$. The two definitions are clearly homeomorphisms, although a formal proof for the definition with $m \geq 0$ would be lengthy. In fact the mapping is piecewise affine, although the region $m \geq 0$ has to be further subdivided to obtain the regions over which the mapping is affine. It remains to show that ϕ has the desired effect on the embedded graph.

Each vertex $\underline{a}_0, \dots, \underline{a}_{n-3}$ lies in P so $q = 0$. Also it lies in all but one of the hyperplanes P_i , and on the same side of the other as the origin, so m is also zero. Hence each of these vertices is fixed. Consider two of these vertices, \underline{a}_i and \underline{a}_j . Then if $n \geq 5$ there is a third vertex \underline{a}_k with $k \leq n - 3$, and both \underline{a}_i and \underline{a}_j belong to P_k . Hence the line segment $\underline{a}_i \underline{a}_j$ also belongs to P_k . Hence $m = q = 0$ over all points of $\underline{a}_i \underline{a}_j$. So these lines are left fixed.

For the vertices $\underline{a}_{n-1}, \underline{a}_{n-2}$, $\underline{p} = \underline{0}$, so $m = d$. A simple calculation shows that \underline{a}_{n-1} and \underline{a}_{n-2} are interchanged.

Now consider the points on the lines $\underline{a}_i \underline{a}_{n-1}$ and

$\underline{a}_i \underline{a}_{n-2}$ for $i = 0, \dots, n-3$. Along these lines m increases linearly from 0 at \underline{a}_i to d at \underline{a}_{n-1} and \underline{a}_{n-2} . Hence the general point on $\underline{a}_i \underline{a}_{n-1}$ has vector

$$\frac{d-m}{d} \underline{a}_i + \frac{mz}{dn} \underline{e},$$

and on $\underline{a}_i \underline{a}_{n-2}$ the general point has vector

$$\frac{d-m}{d} \underline{a}_i + \frac{mz}{d} \underline{e},$$

and these two points are interchanged by the transformation.

Finally a point on $\underline{a}_{n-1} \underline{a}_{n-2}$ has the form

$$(1-k) \frac{z}{n} \underline{e} + kz \underline{e}$$

which is mapped into the point

$$k \frac{z}{n} \underline{e} + (1-k) z \underline{e},$$

the point at the same distance from the other end of the line segment.

Hence ϕ has the desired behaviour and the theorem is proved.

This embedding will of course suffice for any graph of order n : vertices will be interchanged and line segments properly mapped. It is only necessary to select from the automorphisms of K_n defined by products of the above transpositions the automorphisms of the required graph. Hence:

Theorem 4.14.2.

Every graph of order n , with $n \geq 5$ can be homeomorphic symmetrically embedded in R^{n-2} .

There is no evidence that this is the best that can be done except in the cases $n = 5, 6$.

As for other graphs, I have obtained no general results. It is not hard to find a tree which cannot be homeomorphic-symmetrically embedded in the plane: the tree of fig. 4.14.1 cannot. On the other hand, although I have no proof so far, I conjecture that every tree can be homeomorphic-symmetrically embedded in R^3 .

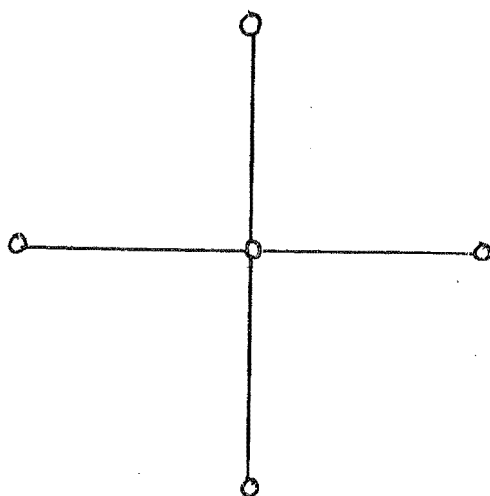


fig. 4.14.1

CHAPTER V

LINEAR SYMMETRIC EMBEDDINGS

5.1 In this chapter we consider symmetric embeddings of finite graphs in R^m when the group of functions is the set of invertible linear transformations. A graph symmetrically embedded for this group will be called linear symmetrically embedded.

As we remarked in section 4.5, there is in general more than one member of the group which induces any given automorphism in the graph. In this case however we can find a simple condition which makes the group Ψ (in the notation of section 4.5) isomorphic to Γ .

Definition 5.1.1.

Let the vertices of an embedding G' of a graph G be $\underline{a}_1, \dots, \underline{a}_n$. Then we define $V_{G'}$, the space spanned by G' , to be the subspace of R^m spanned by $\underline{a}_1, \dots, \underline{a}_n$.

Theorem 5.1.1.

The group Ψ of all invertible linear transformations of R^m inducing automorphisms of G is isomorphic to Γ , the group of G , if and only if the space $V_{G'}$, spanned by G' is the whole of R^m .

Following Theorem 4.5.3, it is necessary only to prove that only the identity linear transformation induces the identity permutation if and only if $V_{G'} = R^m$.

Let M be an invertible linear transformation inducing the identity automorphism. Then for $i = 1, \dots, n$,

$$Ma_r = a_r.$$

Hence every linear combination of a_1, \dots, a_n is mapped into itself. So $V_{G'}$ is held fixed. Then if $V_{G'} = R^m$, M is the identity over R^m .

Suppose on the other hand that $V_{G'}$ is a proper subspace of R^m . Then we can find a basis $\{\underline{u}_1, \dots, \underline{u}_s, \underline{u}_{s+1}, \dots, \underline{u}_m\}$ of R^m such that $\{\underline{u}_1, \dots, \underline{u}_s\}$ is a basis of $V_{G'}$. In the case we are considering s is strictly less than m . We therefore define the following invertible linear transformation M'

$$M'\underline{u}_i = \underline{u}_i \text{ if } i \neq s+1$$

$$M'\underline{u}_{s+1} = -\underline{u}_{s+1}.$$

Then M' induces the identity in G , but is not the identity in R^m .

Clearly if $V_{G'}$ is a proper subspace of R^m , we can linear symmetrically embed G in a space of dimension less than m . The converse is not true: the dimension of $V_{G'}$ is not invariant for all embeddings. Its minimum value is the minimum dimension of a space in which the graph can be linear symmetrically embedded.

Our aim in this chapter is to seek linear symmetric embeddings in which Ψ is isomorphic to Γ , and usually embeddings in spaces of minimum possible dimension. Our result shows that not only are these aims compatible but achieving the second will achieve the first. Unfortunately while it is easy to decide whether or not V_G is the whole space, it is not always easy to decide that there are no linear symmetric embeddings of the graph in a space of smaller dimension.

5.2 Since our aim is to obtain specific embeddings and linear transformations we shall in general consider that there is given in the space a set of orthogonal axes, and we shall represent our linear transformations by their matrices relative to these.

Provided that $V_G = R^m$ the group of matrices is isomorphic with the automorphism group of G . So each matrix is of finite period, equal to the order of the corresponding automorphism. (We use the word period for the least positive integer p such that A^p is the unit matrix rather than order since this already has another meaning for matrices.) Each matrix is therefore invertible, but it need not be orthogonal. We do however have the following construction (theorem 5.2.1). which enables us to obtain an embedding in which all the matrices are orthogonal from any linear symmetric embedding which spans the whole space. Such an embedding is called an

orthogonal symmetric embedding.

First let G' be any embedding of G in R^m with vertices $\underline{a}_1, \dots, \underline{a}_n$, and let A be an automorphism matrix such that, for instance $A\underline{a}_i = \underline{a}_j$.

Now let T be any invertible $m \times m$ matrix, and define $\underline{b}_1, \dots, \underline{b}_n$ by the equation

$$\underline{b}_k = T\underline{a}_k \text{ for each } k = 1, \dots, n.$$

These vertices define another embedding G'' of G in R^m , and the embedding is proper since if x is any real number

$$T((1-x)\underline{a}_i + x\underline{a}_j) = (1-x)\underline{b}_i + x\underline{b}_j.$$

Applying this in the reverse direction, a point can belong to two edges in G'' only if it corresponds to a point belonging to two edges in G' . So if two edges meet other than at a common vertex in G'' they do so in G' , and G' is not a proper embedding. The coincidence of a vertex in G'' with an edge not incident with it in G is dealt with in the same way, and similarly other possible difficulties.

Moreover since $T\underline{a}_i = \underline{b}_i$ and $T\underline{a}_j = \underline{b}_j$,

$$(TAT^{-1})\underline{b}_i = TAT^{-1}(T\underline{a}_i) = T\underline{a}_j = \underline{b}_j.$$

The automorphisms of G are therefore generated in G'' by the matrices like TAT^{-1} : the transformation is a similarity. The embedding G'' is then said to be similar to G' . It is not true that any two linear symmetric embeddings of a graph in the same space are similar.

Theorem 5.2.1.

Given any linear symmetric embedding G' of a finite graph G in R^m which spans R^m there is an orthogonal symmetric embedding of G in R^m similar to G' .

First, if M is any matrix we write the transpose of M as M^T . Then if M is square and invertible we show that $M^T M$ is the matrix of a real positive definite quadratic form. For $M^T M$ is invertible and also symmetric, since $(M^T M)^T = M^T (M^T)^T = M^T M$. And if \underline{y} is any non-zero vector $M\underline{y}$ is non zero, so that $\underline{y}^T (M^T M) \underline{y} = (M\underline{y})^T (M\underline{y}) = \|M\underline{y}\|^2 > 0$.

As G is finite its group is also finite.

Now let ϕ, ψ etc be automorphisms of G and $A(\phi), A(\psi)$ etc the corresponding matrices. We define

$$H = \sum_{\phi} A(\phi)^T A(\phi).$$

From what we have just shown H is a finite sum of positive definite symmetric matrices, so is itself positive definite and symmetric.

Therefore there exists a real invertible matrix P such that $P^T H P = I$, the $m \times m$ identity matrix. But then if ψ is any fixed automorphism,

$$\begin{aligned}
(P^{-1}A(\psi)P)^T(P^{-1}A(\psi)P) &= P^T A(\psi)^T (P^{-1})^T P^{-1} A(\psi) P \\
&= P^T A(\psi)^T H A(\psi) P \\
&= P^T \{A(\psi)^T (\sum_{\phi} A(\phi)^T A(\phi)) A(\psi)\} P \\
&= P^T \{ \sum_{\phi} (A(\phi) A(\psi))^T (A(\phi) A(\psi)) \} P \\
&= P^T \{ \sum_{\phi} A(\phi\psi)^T A(\phi\psi) \} P
\end{aligned}$$

But ψ is fixed, so that as ϕ runs through all automorphisms of G , so does $\phi\psi$. Hence

$$(P^{-1}A(\psi)P)^T(P^{-1}A(\psi)P) = P^T H P = I.$$

So $P^{-1}A(\psi)P$ is orthogonal, and P corresponds to T^{-1} as introduced when defining similar embeddings. We have thus obtained an orthogonal symmetric embedding, with $\underline{b}_1 = P^{-1}\underline{a}_1$, similar to the given embedding.

(The above proof is basically the real version of the theorem for complex transformations given, for instance in (Ne 2 p 244)).

5.3 We have already referred in the previous chapter to linear symmetric embeddings of K_{n+1} in R^n , and our next task is to prove this result.

Theorem 5.3.1.

Let a_0, a_1, \dots, a_n be the $n+1$ vertices of the complete graph K_n . Then the vectors

$$\begin{aligned}
\underline{u}_0 &= (-e_1, -e_2, \dots, -e_n) \\
\underline{u}_1 &= (e_1, -e_2, \dots, -e_n) \\
\underline{u}_2 &= (0, 2e_2, -e_3, \dots, -e_n) \\
&\vdots \\
\underline{u}_i &= (0, \dots, 0, ie_i, -e_{i+1}, \dots, -e_n) \\
&\vdots \\
\underline{u}_n &= (0, \dots, 0, ne_n)
\end{aligned}$$

$$\text{where } e_i = \frac{1}{i} \sqrt{\frac{i(n+1)}{(i+1)n}}$$

form a linear symmetric embedding of K_n in R^m .

In fact this embedding is also orthogonal, and the vertices are all distance 1 from the origin as we shall show.

Let ϕ be any permutation on u_0, \dots, u_n . Then with this embedding the matrix M_ϕ inducing ϕ on the vertices is the product $W_\phi V$ where W_ϕ is the matrix whose i th column is $\underline{a}_{\phi(i)}$ for each $i = 1, \dots, n$ and V is the matrix

$$\begin{bmatrix}
\frac{1}{e_1} & \cdot & \cdot & \cdot & \cdot \\
\frac{1}{2e_1} & \frac{1}{2e_2} & \cdot & \cdot & \cdot \\
\frac{1}{2e_1} & \frac{1}{6e_2} & \frac{1}{3e_3} & \cdot & \cdot \\
\frac{1}{2e_1} & \frac{1}{6e_2} & \frac{1}{12e_3} & \frac{1}{i(i+1)e_i} & \frac{1}{ne_n} \\
\frac{1}{2e_1} & \frac{1}{6e_2} & \frac{1}{12e_3} & \frac{1}{i(i+1)e_i} & \frac{1}{ne_n}
\end{bmatrix}$$

To establish this, we note that V is the inverse of the matrix whose columns are $\underline{a}_1, \dots, \underline{a}_n$, so $V\underline{a}_i$ is the

column vector whose i th element is 1 and which has zero elsewhere. If this vector is \underline{v}_i , then $W_\phi \underline{v}_i$ is the i th column of W_ϕ , i.e. $\underline{a}_\phi(i)$.

There remains the vertex \underline{a}_0 . Inspecting the vectors we see that

$$\underline{a}_0 + \underline{a}_1 + \dots + \underline{a}_n = \underline{0}.$$

Hence

$$\begin{aligned} M_\phi \underline{a}_0 &= -M_\phi (\underline{a}_1 + \underline{a}_2 + \dots + \underline{a}_n) \\ &= -M_\phi \underline{a}_1 - M_\phi \underline{a}_2 - \dots - M_\phi \underline{a}_n \\ &= -\underline{a}_\phi(1) - \underline{a}_\phi(2) - \dots - \underline{a}_\phi(n). \end{aligned}$$

But as also

$$\underline{a}_\phi(0) + \underline{a}_\phi(1) + \dots + \underline{a}_\phi(n) = \underline{0}.$$

$$M_\phi \underline{a}_0 = \underline{a}_\phi(0).$$

The n vectors $\underline{a}_1, \dots, \underline{a}_n$ are certainly linearly independent so span R^n , so V_G is R^n . Thus the matrices do form a group isomorphic to the desired group S_{n+1} .

We have not shown that the embedding is proper. But it can be shown that the set of vectors $\underline{a}_0, \dots, \underline{a}_n$ has dimension n , and that further any set of less than n of them is linearly independent. Therefore no four are coplanar, and in a finite graph this is enough to ensure that the embedding is proper.

We define the inner product of the vectors $\underline{u} = (u_1, u_2, \dots, u_n)$ and $\underline{v} = (v_1, v_2, \dots, v_n)$ in the usual way as $u_1 v_1 + u_2 v_2 + \dots + u_n v_n$. Then the product of any two

different vectors $\underline{a}_i, \underline{a}_j$ with $0 \leq i < j \leq n$ is $-\frac{1}{n}$.

$$\begin{aligned} \text{For } (\underline{a}_i, \underline{a}_j) &= -je_j^2 + e_{j+1}^2 + \dots + e_n^2 \\ &= -j \cdot \frac{1}{j^2} \frac{j(n+1)}{(j+1)n} + \dots + \frac{1}{n^2} \\ &= \frac{n+1}{n} \left\{ -\frac{1}{j+1} + \sum_{k=j+1}^n \frac{1}{k(k+1)} \right\} \end{aligned}$$

$$\text{But } \frac{1}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1}, \quad \text{so}$$

$$\sum_{k=j+1}^n \frac{1}{k(k+1)} = \frac{1}{j+1} - \frac{1}{n+1} \quad \text{and}$$

$$(\underline{a}_i, \underline{a}_j) = \frac{n+1}{n} \left\{ -\frac{1}{j+1} + \frac{1}{j+1} - \frac{1}{n+1} \right\} = -\frac{1}{n},$$

And the inner product of \underline{a}_i with itself:

$$\begin{aligned} (\underline{a}_i, \underline{a}_i) &= i^2 e_i^2 + e_{i+1}^2 + \dots + e_n^2 \\ &= \frac{n+1}{n} \left\{ \frac{i}{i+1} + \sum_{k=i+1}^n \frac{1}{k(k+1)} \right\} \\ &= \frac{n+1}{n} \cdot \left\{ \frac{i}{i+1} + \frac{i}{i+1} - \frac{1}{n+1} \right\} \\ &= 1, \end{aligned}$$

showing that the length of each is 1.

These results are sufficient to imply orthogonality.

As a corollary we deduce directly:

Theorem 5.3.2.

Every graph of order n may be linear symmetrically embedded in R^{n-1} .

And again from this and Theorem 5.2.1,

Theorem 5.3.3.

Every graph of order n may be orthogonal symmetrically embedded in R^{n-1} .

5.4 There is also an alternative linear symmetric embedding of K_{n+1} in R^n with the vectors

$$\begin{aligned}\underline{a}_0 &= \left(-\frac{1}{n}, -\frac{1}{n}, \dots, -\frac{1}{n}\right) \\ \underline{a}_1 &= (1, 0, \dots, 0) \\ \underline{a}_2 &= (0, 1, \dots, 0) \\ &\vdots \\ \underline{a}_n &= (0, \dots, 0, 1).\end{aligned}$$

The automorphism matrices are also simpler, the columns of $A(\phi)$ being simply $\underline{a}_{\phi(i)}$ for $i = 1, \dots, n$. This embedding suffers however from not being orthogonal.

Applying the technique of theorem 5.2.1 does not seem to yield any simple natural orthogonalisation: when $n = 2$

the matrix H is $\begin{bmatrix} 5 & -2 \\ -2 & 5 \end{bmatrix}$.

An even simpler embedding can be found of K_n in R^n :

$$\begin{aligned}\underline{a}_1 &= (1, 0, \dots, 0) \\ \underline{a}_2 &= (0, 1, \dots, 0) \\ &\vdots \\ \underline{a}_n &= (0, 0, \dots, 1)\end{aligned}$$

5.5 We made use in theorem 5.3.1 of the fact that

$$\underline{a}_0 + \underline{a}_1 + \dots + \underline{a}_n = \underline{0}$$

for the particular embedding used there. It is frequently useful to embed a graph so that this relationship holds as we shall subsequently show. We begin with a standard definition.

Definition 5.5.1.

Let $\underline{u}_1, \dots, \underline{u}_n$ be the position vectors in a real coordinate space R^m . Then the centroid $\bar{\underline{u}}$ of $\underline{u}_1, \dots, \underline{u}_n$ is the point

$$\bar{\underline{u}} = \frac{1}{n} (\underline{u}_1 + \underline{u}_2 + \dots + \underline{u}_n).$$

Theorem 5.5.1.

Let $\underline{a}_1, \dots, \underline{a}_n$ be the vertices of an embedding of a graph G in the real space R^m . Then the centroid of $\underline{a}_1, \dots, \underline{a}_n$ is held fixed by all the automorphism-inducing linear transformations of the space.

For let A be any automorphism, and $\bar{\underline{a}}$ the centroid.

Then

$$\begin{aligned} A\bar{\underline{a}} &= A\left(\frac{1}{n}\underline{a}_1 + \frac{1}{n}\underline{a}_2 + \dots + \frac{1}{n}\underline{a}_n\right) \\ &= \frac{1}{n}A\underline{a}_1 + \frac{1}{n}A\underline{a}_2 + \dots + \frac{1}{n}A\underline{a}_n. \end{aligned}$$

But $A\underline{a}_1, A\underline{a}_2, \dots, A\underline{a}_n$ are simply $\underline{a}_1, \underline{a}_2, \dots, \underline{a}_n$ reordered.

$$\text{So } A\bar{\underline{a}} = \frac{1}{n}\underline{a}_1 + \frac{1}{n}\underline{a}_2 + \dots + \frac{1}{n}\underline{a}_n = \bar{\underline{a}}.$$

The theorem may be extended to apply to several other sets of vertices, for instance the centroid of any

transitivity class is also held fixed.

Theorem 5.5.2.

If a graph G possesses a linear symmetric embedding G' in the real space R^m , G has a linear symmetric embedding in R^m with the centroid of the vertices of G at the origin, and with matrices the same as those for the embedding G' .

Let $\underline{a}_1, \dots, \underline{a}_n$ be the position vectors of the vertices of G' . Then the centroid of the vertices of G' is

$$\underline{\bar{a}} = \frac{1}{n} \sum_{i=1}^n \underline{a}_i$$

Now define $\underline{b}_i = \underline{a}_i - \underline{\bar{a}}$.

The centroid of $\underline{b}_1, \dots, \underline{b}_n$ is

$$\begin{aligned} \underline{\bar{b}} &= \frac{1}{n} \sum_{i=1}^n \underline{b}_i = \frac{1}{n} \sum_{i=1}^n (\underline{a}_i - \underline{\bar{a}}) \\ &= \frac{1}{n} (\sum \underline{a}_i - n\underline{\bar{a}}) \\ &= \underline{0}. \end{aligned}$$

And if A is an automorphism-inducing matrix with, for example $A\underline{a}_i = \underline{a}_j$, then

$$\begin{aligned} A\underline{b}_i &= A(\underline{a}_i - \underline{\bar{a}}) \\ &= A\underline{a}_i - A\underline{\bar{a}} \\ &= \underline{a}_j - \underline{\bar{a}} \\ &= \underline{b}_j. \end{aligned}$$

So A is also an automorphism-inducing matrix for the new embedding.

Since the centroid must be left fixed it is often useful to embed a graph which has a single fixed point with that point at the centroid. If the centroid can also be put at the origin this may save a dimension in the space R^m .

It may not however be possible to embed this fixed vertex at the centroid: consider for instance the graph with three vertices a_1, a_2, a_3 and the single edge (a_1, a_2) . If \underline{a}_3 is the centroid of $\underline{a}_1, \underline{a}_2$ and \underline{a}_3 it must be half-way between \underline{a}_1 and \underline{a}_2 . But this would place it in the line segment $\underline{a}_1 \underline{a}_2$.

5.6 The rest of the chapter will be concerned with demonstrating methods of obtaining linear symmetric embeddings for given graphs and with conditions for a graph to possess a linear symmetric embedding in any given space.

In a sense this work is unnecessary, since we know that any graph of order n can be linear symmetrically embedded in a space of dimension $n-1$, and that especially if the embedding in R^n given at the end of section 5.4 is used, the matrix elements can be made simple.

Yet there is more to be said. This embedding is frequently very inefficient as far as dimension is concerned. Nor have we yet demonstrated that our theorem 5.3.1 is the best that can be done.

One obvious condition for the possibility of embedding a graph G with group Γ in a space R^m is that R^m must possess

a group of linear transformations - indeed of orthogonal transformations - isomorphic to Γ . In R^2 and R^3 where all these groups are well-known this can be a useful check when Γ is known.

5.7 The concepts of determining set and degree of freedom were introduced in section 2.13 specifically for the part they play in constructing linear symmetric embeddings.

Theorem 5.7.1.

Let G be a graph linear symmetrically embedded in R^m . If $\{a_1, a_2, \dots, a_n\}$ is a minimal determining set of G , the vectors $\underline{a}_1, \underline{a}_2, \dots, \underline{a}_n$ are linearly independent.

For if they are not linearly independent we can express some member, say \underline{a}_n , in terms of the others:

$$\underline{a}_n = \alpha_1 \underline{a}_1 + \alpha_2 \underline{a}_2 + \dots + \alpha_{n-1} \underline{a}_{n-1}$$

for suitable scalars $\alpha_1, \dots, \alpha_{n-1}$.

Then if ϕ is any automorphism and Φ its matrix,

$$\Phi \underline{a}_n = \alpha_1 \Phi \underline{a}_1 + \alpha_2 \Phi \underline{a}_2 + \dots + \alpha_{n-1} \Phi \underline{a}_{n-1}.$$

Thus $\Phi \underline{a}_n$ is determined by $\Phi \underline{a}_1, \dots, \Phi \underline{a}_{n-1}$, and this implies that $\phi(a_n)$ is determined by $\phi(a_1), \dots, \phi(a_{n-1})$. Hence $\{a_1, \dots, a_n\}$ is not a minimal determining set.

From this theorem we deduce immediately

Theorem 5.7.2.

If G' is a linear symmetric embedding of a graph G with degree of freedom n in the real space R^m , then $n \leq m$.

This gives a lower bound to the possible dimension for linear symmetrically embedding a graph, but this bound is not always attained.

As the complete graph K_n on n vertices has degree of freedom $n-1$, the result of Theorem 5.3.1 is the best that can be obtained.

Given any real coordinate space we can construct a graph with only one degree of freedom which cannot be linear symmetrically embedded in any proper subspace. Such a graph is the Frucht graph of section 3.3 for the symmetric group S_{n+1} , which cannot be linear symmetrically embedded in any space of dimension less than n . All the Frucht graphs have one degree of freedom.

5.8 As far as symmetry is concerned a graph and its complement are the same: an automorphism of one is an automorphism of the other. But an embedding of one may not be an embedding of the other. For instance, the complement of a planar graph need not be planar. Thus for $m = 1$ or 2 the statement that the graph G can be linear symmetrically embedded in R^m does not prove that its complement \bar{G} can be embedded in R^m at all.

And even if it can be so embedded there is no guarantee that it can be linear symmetrically embedded in the space. The graph on three vertices a_1, a_2, a_3 with the two edges (a_1, a_2) and (a_2, a_3) has the linear symmetric embedding $\underline{a}_1 = (-1)$, $\underline{a}_2 = (0)$, $\underline{a}_3 = (1)$ in R^1 , but its complement with only the edge (a_1, a_3) cannot be linearly symmetrically embedded in R^1 .

Hence no criterion which relies solely on isomorphism invariants either of the individual vertices or the graph as a whole, which coincide for a graph and its complement, can decide the actual minimum dimension of a real space in which the graph can be linear symmetrically embedded.

5.9 In the next two sections we discuss aspects of the actual process of constructing embeddings and follow this in section 5.11 with an actual example.

First and most simply consider a graph G which has components G_1, G_2, \dots, G_c , each component G_i having a linear symmetric embedding G_i' in R^{r_i} . Then we can form a linear symmetric embedding G' of G in R^r where $r = r_1 + r_2 + \dots + r_c$.

We consider the vertices in each component in turn. A vertex in G_1 is represented by a vector in R^r whose first r_1 components are those of the corresponding vertex of G_1' in R^{r_1} , and the rest zero. Then a vertex in G_2 is represented by a vector whose first r_1 components are zero, the next r_2 are the components of the corresponding vector

in G_2^1 , and finally the rest again zero. And so for all the components.

Two conditions must however be watched. Firstly the zero of R^r corresponds to all the zeros in the individual embeddings. Hence zero must not be a vertex or a point on a line segment in more than one of the embeddings G_1^1, \dots, G_c^1 . Secondly it must be possible to interchange isomorphic components in G' by a suitable linear transformation. Two isomorphic components must therefore have similar embeddings (in the sense of section 5.2).

Since any graph with n vertices can be linear symmetrically embedded in R^{n-1} , if component G_i has n_i vertices, $r_i \leq n_i - 1$. Summing over all the components, $r_1 + r_2 + \dots + r_c \leq (n_1 + n_2 + \dots + n_c) - c$. Hence if n is the order of G , the minimum dimension r for any linear symmetric embedding is governed by

$$r \leq n - c.$$

It must however be noted that this does not hold if there is more than one component of one or two vertices, in line with the first exception above. For in either of these cases the origin is a point of the embedding.

In special cases the necessary dimension can of course often be reduced. For instance suppose G has c isomorphic asymmetric components, with $c \geq 2$. Then if each component is planar, G can be embedded linear symmetrically

in R^C , and if G is not planar the space required is R^{C+1} .

5.10 It is possible to do something of the same sort with transitivity classes, reserving a subspace of suitable dimension for each class. Again the condition about vertices at the origin or edges passing through that point complicates any general condition. The process is here simply an aid to tidiness in the embedding.

Space may also be saved by considering dependent transitivity classes. Let C be a transitivity class in a graph G . Then the automorphism closure cC may also include another transitivity class C' , in which case C' is said to be dependent on C . Two classes may of course depend on each other, as in fig. 5.10.1. When one class depends on another some amalgamation of spaces is possible.

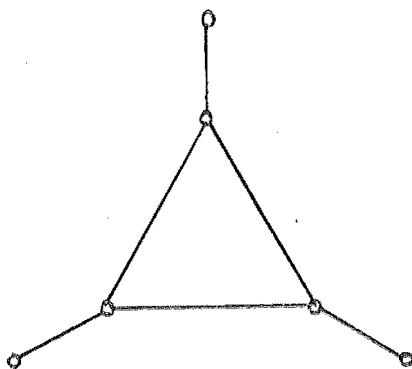


fig. 5.10.1.

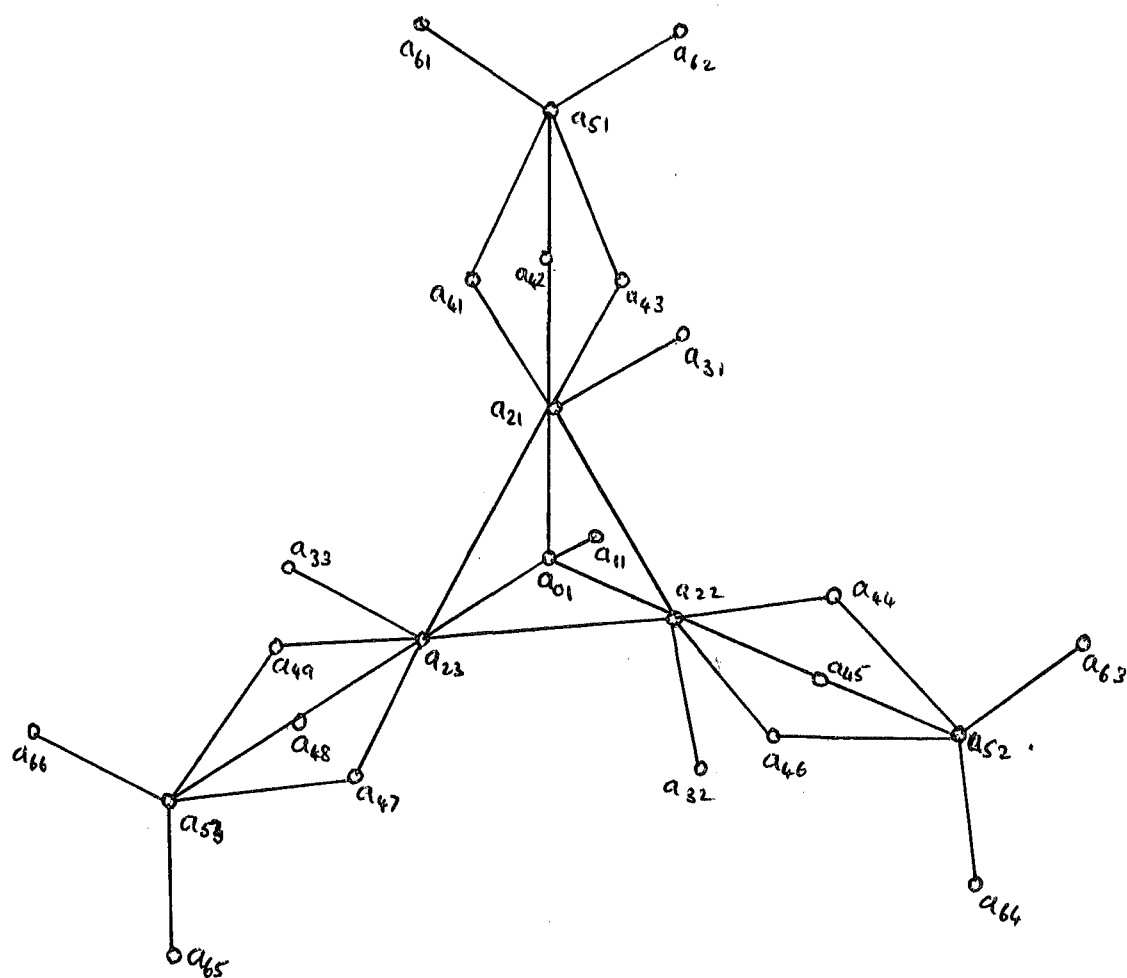


fig. 5.11.1

5.11 We illustrate one method of constructing linear symmetric embeddings with reasonable efficiency on the dimension of the space with the graph in fig. 5.11.1.

The first step is to sort out the transitivity classes. This has already been done in labelling the figure: the first suffix for each vertex gives the transitivity class. There are seven classes, two of which contain only a single vertex.

It is not strictly necessary to investigate the degree of freedom, but it does give a lower bound to the possible dimension of an embedding. In each of the triples $\{a_{41}, a_{42}, a_{43}\}, \{a_{44}, a_{45}, a_{46}\}, \{a_{47}, a_{48}, a_{49}\}$ two vertices must be fixed irrespective of what happens elsewhere. And in each of the pairs $\{a_{61}, a_{62}\}, \{a_{63}, a_{64}\}, \{a_{65}, a_{66}\}$, one must be fixed. So every determining set must contain at least nine such vertices, such as

$$\{a_{41}, a_{42}, a_{61}, a_{44}, a_{45}, a_{63}, a_{47}, a_{48}, a_{65}\}.$$

But this is a determining set, so the degree of freedom is 9.

In the process we use here we build up the embedding, adding a class at a time. We begin with the triangle $a_{21}a_{22}a_{23}$, which we embed in a plane with two orthogonal unit vectors \underline{e}_1 and \underline{e}_2 and the position vectors

$$\underline{a}_{21} = \underline{e}_1, \underline{a}_{22} = -\frac{1}{2}\underline{e}_1 + \frac{\sqrt{3}}{2}\underline{e}_2, \underline{a}_{23} = -\frac{1}{2}\underline{e}_1 - \frac{\sqrt{3}}{2}\underline{e}_2.$$

(We shall use the method of defining new unit vectors, orthogonal to all the previous unit vectors, as they may

be necessary since we do not yet know the dimension of the necessary space. They may be replaced as a final step by the coordinate vector with 1 in the i th place and 0 elsewhere for \underline{e}_i .)

We can also place the fixed vertex a_{01} at the origin.

Considering next the class C_5 , it is dependent on the class C_2 already embedded. We may therefore take $\underline{a}_{5i} = k\underline{a}_{2i}$ for each $i = 1, 2, 3$ and k some real number greater than 1. We take $k = 3$ for convenience. Then

$$\underline{a}_{51} = 3\underline{e}_1, \underline{a}_{52} = -\frac{3}{2}\underline{e}_1 + \frac{3\sqrt{3}}{2}\underline{e}_2, \underline{a}_{53} = -\frac{3}{2}\underline{e}_1 - \frac{3\sqrt{3}}{2}\underline{e}_2.$$

Now as the triplets $\{a_{41}, a_{42}, a_{43}\}, \{a_{44}, a_{45}, a_{46}\}, \{a_{47}, a_{48}, a_{49}\}$ can be permuted independently a pair of orthogonal unit vectors will be required for each. We embed each triplet as an equilateral triangle with its centroid on the ray through \underline{a}_{2i} and \underline{a}_{5i} . We take the centroids at $2\underline{a}_{21}, 2\underline{a}_{22}$ and $2\underline{a}_{23}$ for convenience.

$$\underline{a}_{41} = 2\underline{e}_1 + \underline{e}_3, \underline{a}_{42} = 2\underline{e}_1 - \frac{1}{2}\underline{e}_3 + \frac{\sqrt{3}}{2}\underline{e}_4, \underline{a}_{43} = 2\underline{e}_1 - \frac{1}{2}\underline{e}_3 - \frac{\sqrt{3}}{2}\underline{e}_4$$

$$\underline{a}_{44} = -\underline{e}_1 + \frac{\sqrt{3}}{2}\underline{e}_2 + \underline{e}_5, \underline{a}_{45} = -\underline{e}_1 + \frac{\sqrt{3}}{2}\underline{e}_2 - \frac{1}{2}\underline{e}_5 + \frac{\sqrt{3}}{2}\underline{e}_6$$

$$\underline{a}_{46} = -\underline{e}_1 + \sqrt{3}\underline{e}_2 - \frac{1}{2}\underline{e}_5 - \frac{\sqrt{3}}{2}\underline{e}_6$$

$$\underline{a}_{47} = -\underline{e}_1 - \sqrt{3}\underline{e}_2 + \underline{e}_7, \underline{a}_{48} = -\underline{e}_1 - \sqrt{3}\underline{e}_2 - \frac{1}{2}\underline{e}_7 + \frac{\sqrt{3}}{2}\underline{e}_8,$$

$$\underline{a}_{49} = -\underline{e}_1 - \sqrt{3}\underline{e}_2 - \frac{1}{2}\underline{e}_7 - \frac{\sqrt{3}}{2}\underline{e}_8.$$

The pairs of class C_6 raise a similar problem, and a single new unit vector, orthogonal to all those previously

defined is added for each of these pairs. We take $\underline{a}_{6,2i-1}$ and $\underline{a}_{6,2i}$ on a line through $\underline{a}_{5,i}$ parallel to \underline{e}_{8+i} , one on each side of \underline{a}_{5i} . Thus

$$\begin{aligned}\underline{a}_{61} &= 3\underline{e}_1 + \underline{e}_9, & \underline{a}_{62} &= 3\underline{e}_1 - \underline{e}_9 \\ \underline{a}_{63} &= -\frac{3}{2}\underline{e}_1 + \frac{3\sqrt{3}}{2}\underline{e}_2 + \underline{e}_{10}, & \underline{a}_{64} &= -\frac{3}{2}\underline{e}_1 + \frac{3\sqrt{3}}{2}\underline{e}_2 - \underline{e}_{10} \\ \underline{a}_{65} &= -\frac{3}{2}\underline{e}_1 - \frac{3\sqrt{3}}{2}\underline{e}_2 + \underline{e}_{11}, & \underline{a}_{66} &= -\frac{3}{2}\underline{e}_1 - \frac{3\sqrt{3}}{2}\underline{e}_2 - \underline{e}_{11}.\end{aligned}$$

There remain the members of classes C_1 and C_3 . A single further unit vector \underline{e}_{12} will suffice take care of them:

$$\begin{aligned}\underline{a}_{11} &= \underline{e}_{12} \\ \underline{a}_{31} &= \underline{a}_{21} + \underline{e}_{12} = \underline{e}_1 + \underline{e}_{12} \\ \underline{a}_{32} &= \underline{a}_{22} + \underline{e}_{12} = -\frac{1}{2}\underline{e}_1 + \frac{\sqrt{3}}{2}\underline{e}_2 + \underline{e}_{12} \\ \underline{a}_{33} &= \underline{a}_{23} + \underline{e}_{12} = -\frac{1}{2}\underline{e}_1 - \frac{\sqrt{3}}{2}\underline{e}_2 + \underline{e}_{12}\end{aligned}$$

We note that the graph has a linear symmetric embedding in R^{12} , dimension 3 more than the degree of freedom. It may well be possible to find an embedding in a space of smaller dimension but this embedding has the advantage that no vertex position vector involves more than four of the unit vectors $\underline{e}_1, \dots, \underline{e}_{12}$. The matrices for a set of generators are also fairly simple.

5.12 Finally in this chapter we consider the following problem. Suppose we have a finite group Γ . What is the minimum dimension of a space R^m in which there can be linear symmetrically embedded a graph with group isomorphic

to Γ ? We will not attempt to find this minimum but content ourselves with the following theorem.

Theorem 5.12.1.

Let Γ be a group of finite order n . Then there is a Frucht graph of Γ which can be linear symmetrically embedded in \mathbb{R}^n if Γ has a set of generators with no elements of order 2.

Starting with Γ we form the Cayley colour group of Γ , taking its vertices as a_1, a_2, \dots, a_n . We embed these points in \mathbb{R}^n as

$$\begin{aligned}\underline{a}_1 &= (1, 0, 0, \dots, 0) \\ \underline{a}_2 &= (0, 1, 0, \dots, 0) \\ &\vdots \\ \underline{a}_n &= (0, 0, 0, \dots, 1)\end{aligned}$$

If we attempt to embed the colour group with these vertices we find that the pair of directed lines $\underline{a}_i \underline{a}_j$ and $\underline{a}_j \underline{a}_i$ join \underline{a}_j and \underline{a}_i . But if Γ has a set of generators with none of order 2 and we omit all the other directed lines, there is at most one directed line between \underline{a}_i and \underline{a}_j .

Now each pair of vertices defines a unique plane through the origin and two planes corresponding to disjoint pairs meet only in the origin. If we therefore replace the directed line segment $\underline{a}_i \underline{a}_j$ if it corresponds to the generator γ_k by a subgraph like that in fig. 5.12.1, we obtain a linear symmetric embedding of the Frucht graph in \mathbb{R}^n .

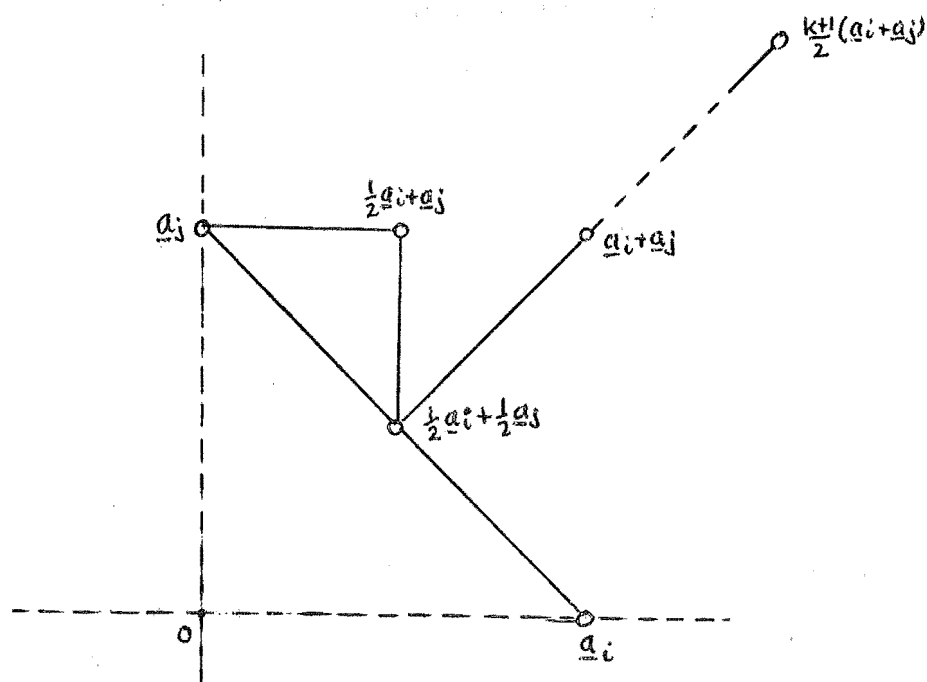


fig. 5.12.1.

In this embedding all the matrices are permutation matrices, that is, each matrix can be obtained from the identity matrix by permuting the rows.

This is probably not the best that can be done. I conjecture as follows:

Conjecture 5.12.1.

The restriction on elements of order 2 in theorem 5.12.1 may be removed.

Conjecture 5.12.2.

The dimension may be reduced to $n-1$.

The first conjecture would require a different subgraph in a different plane, at least for the elements of

order 2. The second would involve the replacement of the vectors of $\underline{a}_1, \dots, \underline{a}_n$ by vectors of the type used in theorem 5.3.1. The matrices are also less simple.

CHAPTER VI

EMBEDDING INFINITE GRAPHS

6.1 Our treatment of linear symmetric embeddings in the previous chapter was concerned wholly with finite graphs, and cannot in every respect be generalised to infinite graphs. The first part of this chapter considers the problem of a linear symmetric embedding for infinite graphs, showing that there may not always be such an embedding. Next we define the affine symmetric embeddings in section 6.5 and submit these to an investigation of the necessary dimension.

We stated in theorem 4.1.1 conditions for an embedding of a graph: two of these have particular relevance for an infinite graph: each vertex must be of finite valency, and the set of vertices must be countable. These conditions will always be assumed in this chapter.

6.2 The infinite graphs with finite groups are the most like the finite graphs in their linear symmetric embedding properties. Each transitivity class is necessarily finite and the degree of freedom is also finite, being the number of factors in some factorisation of the order of the group (theorem 2.13.1).

It would appear that every such graph can be linear symmetrically embedded in some coordinate space of sufficient dimensions: once a certain finite number of transitivity classes has been embedded, including a determining set of vertices, the remaining vertices must simply be suitably placed in the space already determined. There is no difficulty in doing this in any particular case but a general method is more difficult to state and justify.

6.3 If an infinite graph has an infinite group then it still need not have any infinite transitivity classes.

Theorem 6.3.1.

If an infinite graph has one or more infinite transitivity classes it cannot be orthogonal symmetrically embedded in any real coordinate space.

For if a graph be orthogonal symmetrically embedded in a real coordinate space all the vertices of a given transitivity class are at the same distance from the origin. But if any infinite set is bounded in such a space it possesses a limit point, violating the topological considerations in Definition 4.1.1.

According to theorem 5.2.1 for finite groups this would imply that a graph with infinite transitivity classes had no linear symmetric embedding. But the theorem does not apply to these graphs since the construction involves

infinite transitivity classes it has an infinite degree of freedom.

For suppose that a graph has only a finite degree of freedom and only finite transitivity classes. Then it has a finite determining set, whose members have each only a finite number of images. Adding members of the determining set in turn we obtain a product of a finite number of finite numbers as the order of the group as in theorem 2.13.1. The group is therefore not infinite, contrary to hypothesis.

Theorem 6.4.2.

An infinite graph with an infinite group but no infinite transitivity classes cannot be linear symmetrically embedded in any real coordinate space.

For the graph has an infinite degree of freedom and an extension of theorem 5.6.1 gives the result immediately.

6.5 Since so many infinite graphs cannot be linear symmetrically embedded in any real coordinate space we look for another class of convenient functions which will as nearly as possible be to the infinite graphs what the linear transformations are to the finite graphs.

Such a class is the collection of invertible affine transformations. Since we are concerned directly with coordinate representations, we define these directly in terms of linear transformations and vectors.

Definition 6.5.1.

An affine transformation of the space R^m onto itself is a function of the type

$$\alpha(\underline{x}) = A\underline{x} + \underline{a}$$

where A is a linear transformation of R^m onto itself and \underline{a} is a constant vector.

These transformations are precisely those which send every pair of parallel hyperplanes into a pair of parallel hyperplanes. Given two proper m -simplexes in R^m there is precisely one (invertible) affine transformation of R^m onto itself which maps one simplex onto the other.

If α and β are two affine transformations, with

$$\alpha(\underline{x}) = A\underline{x} + \underline{a} \quad \text{and} \quad \beta(\underline{x}) = B\underline{x} + \underline{b},$$

$$\text{then } \beta\alpha(\underline{x}) = \beta(\alpha(\underline{x})) = BA\underline{x} + (B\underline{a} + \underline{b}).$$

Hence the product of two affine transformations is an affine transformation, and there is an inverse to α if and only if A is invertible. We have in fact that

$$\alpha^{-1}(\underline{x}) = A^{-1}\underline{x} - A^{-1}\underline{a}.$$

We also notice that the mapping from the group of invertible affine transformations onto the group of invertible linear transformations is a homomorphism.

Since every linear transformation is affine, every linear symmetric embedding is an affine symmetric embedding.

And every affine symmetric embedding is a piecewise affine symmetric embedding.

6.6 We have already seen that even if there exists a linear symmetric embedding of an infinite graph it will not in general be possible to apply the technique of section 5.2 to obtain an orthogonal symmetric embedding. There is for affine transformations a subgroup of orthogonal affine transformations, which may be conveniently characterised for our purposes by saying that their matrix part is orthogonal. Many graphs which do not possess orthogonal (linear) symmetric embeddings do possess orthogonal affine symmetric embeddings. A necessary and sufficient condition for the process of section 5.2 to work is that the number of matrices involved be finite. This condition will naturally be satisfied if the graph has a finite group, but it may also be satisfied when there is an infinite group. In this case the homomorphism from the group of affine transformations to the group of their linear parts has an infinite kernel. This kernel consists of those affine transformations which have the identity linear transformation part: they are of the form

$$\alpha(\underline{x}) = \underline{x} + \underline{a}.$$

To these is given the name translation. As any non-trivial group of translations is necessarily infinite, we may state that the orthogonalisation process can be applied to an

affine symmetric embedding of an infinite graph only if either the group is finite or the embedded graph has at least one translation among its affine transformations.

We cannot always obtain orthogonal affine symmetric embedding similar to any given affine symmetric embedding, as is shown in fig. 6.3.1. But this graph does also have an orthogonal affine symmetric embedding in which the matrices are all either

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix},$$

as in fig. 6.6.1. I make the following conjecture :

Conjecture 6.6.1.

Every graph G which has an affine symmetric embedding in a space \mathbb{R}^m has an orthogonal affine symmetric embedding in that space.

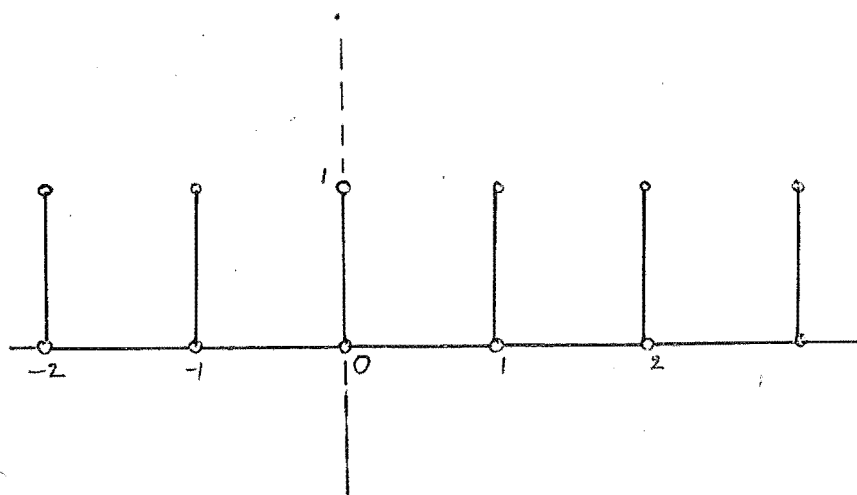


fig. 6.6.1.

6.7 We showed for linear symmetric embeddings that the dimension of any space in which the graph was embedded was at least equal to the degree of freedom (theorem 5.7.2). But it is easy to find infinite graphs affine symmetrically embedded in spaces of dimension less than the degree of freedom. One example is the graph with vertices a_n and edges (a_n, a_{n+1}) for all integers n . This graph can be affine symmetrically embedded in R^1 with $\underline{a}_n = (n)$. But its degree of freedom is 2.

But we do have a theorem on the same lines as theorem 5.7.2:

Theorem 6.7.1.

If the graph G has an affine symmetric embedding G' in the real space R^m and G has degree of freedom n , then $m \geq n - 1$.

Let a_0, \dots, a_{m+1} be a set of vertices in G , with $\underline{a}_0, \dots, \underline{a}_{m+1}$ the corresponding points in G' . Each affine transformation on R^m induces a linear transformation on the $m+1$ differences $(\underline{a}_1 - \underline{a}_0), (\underline{a}_2 - \underline{a}_0), \dots, (\underline{a}_{m+1} - \underline{a}_0)$. As the space of differences is of dimension m , these vectors are linearly dependent, and without loss of generality we may write

$$(\underline{a}_{m+1} - \underline{a}_0) = k_1(\underline{a}_1 - \underline{a}_0) + k_2(\underline{a}_2 - \underline{a}_0) + \dots + k_m(\underline{a}_m - \underline{a}_0)$$

for suitable real numbers k_1, k_2, \dots, k_m .

Now let $\beta(\underline{x}) = B\underline{x} + b$ be an affine transformation in R^m generating an automorphism of G . Then the induced linear transformation β' has $\beta'(\underline{a}_1 - \underline{a}_0) = B(\underline{a}_1 - \underline{a}_0)$. Whence we calculate that

$$B\underline{a}_{m+1} = k_1 B\underline{a}_1 + k_2 B\underline{a}_2 + \dots + k_m B\underline{a}_m - (k_1 + k_2 + \dots + k_m - 1) B\underline{a}_0,$$

which in turn implies that

$$B\underline{a}_{m+1} + b = k_1 (B\underline{a}_1 + b) + \dots + k_m (B\underline{a}_m + b) - (k_1 + \dots + k_m - 1) (B\underline{a}_0 + b);$$

that is

$$\beta(\underline{a}_{m+1}) = k_1 \beta(\underline{a}_1) + \dots + k_m \beta(\underline{a}_m) - (k_1 + \dots + k_m - 1) \beta(\underline{a}_0).$$

Hence the image of \underline{a}_{m+1} is determined completely by the images of $\underline{a}_0, \dots, \underline{a}_m$, and thus $\underline{a}_0, \dots, \underline{a}_{m+1}$ is not a minimal determining set. The degree of freedom n is thus at most $m+1$.

We deduce immediately as a corollary:

Theorem 6.7.2.

No graph with an infinite degree of freedom can be affine symmetrically embedded in any real space R^m .

There certainly are graphs with an infinite degree of freedom. The null graph on a countable set of vertices is one such.

6.8 We may consider which groups have infinite graphs which may be affine symmetrically embedded in some suitable real space. In particular we consider the possibility of embedding the corresponding Frucht graph.

The condition that the number of vertices must be countable for an embedding to exist at all eliminates all non-countable groups. The condition that there may only be a finite number of edges incident with a given vertex obliges us to restrict our attention to finitely generated groups and to the reduced form of the Frucht graph which retains the edges only for a set of generators. For abelian groups we can then prove the existence of such an embedding:

Theorem 6.8.1.

Every finitely generated abelian group has a Frucht graph which may be affinely symmetrically embedded in a suitable real space R^m .

Every finitely generated abelian group can be expressed as the direct product (sum in additive notation) of a finite number of cyclic groups of finite or infinite order. (See for example (Le 1) p 151.) The number of cyclic groups of infinite order in any decomposition is an invariant (the Betti number). There are also invariants of the finite terms of the decomposition but these do not concern us here.

Let the group be of the form

$$Z_1 \otimes Z_2 \otimes \dots \otimes Z_r \otimes C_1 \otimes \dots \otimes C_s$$

where Z_1, \dots, Z_r are infinite cyclic groups and C_1, \dots, C_s are finite cyclic groups with C_i of order n_i . Then we embed first the Cayley colour group in R^{r+2s+1} with the element $(\zeta_1, \zeta_2, \dots, \zeta_r, \gamma_1, \dots, \gamma_s)$ represented by the point

$$(0, \zeta_1, \zeta_2, \dots, \zeta_r, p_1, q_1, p_2, q_2, \dots, p_s, q_s)$$

where

$$p_i = \cos \frac{2\pi\gamma_i}{n_i}, \quad q_i = \sin \frac{2\pi\gamma_i}{n_i}.$$

(We have here committed a slight abuse of the notation. For the infinite cyclic groups ζ_i is used both for the member of Z_i and for the integer corresponding to that member under some isomorphism onto the integers. For the groups of finite order the symbol γ_i is used both for the member of C_i and for any integer corresponding to it in the homomorphism from the integers onto C_i).

There will be a directed line segment between $(\zeta_1, \dots, \zeta_r, \gamma_1, \dots, \gamma_s)$ and $(\zeta'_1, \dots, \zeta'_r, \gamma'_1, \dots, \gamma'_s)$ in the reduced Cayley colour group (with the line segments corresponding to the appropriate generators only retained) only if the two group elements differ only in one place and then only by a single power of the corresponding generator. We replace the coloured line segments by the appropriate subgraphs (as defined in section 3.3) lying in planes as defined below.

For the infinite generators the plane is that defined by the so far un-used first component and the component assigned to the generator. For the finite generators the plane is that already assigned to the generator. The first component has no other purpose, and may be omitted when there are no infinite generators. The graph then becomes

finite and the embedding linear symmetric. The first component may indeed be dispensed with in all cases other than the infinite cyclic group itself.

With the embedding given above the generating transformations take simple forms: for the generator of Z_i a translation $\tau_i(x) = \underline{x} + \underline{e}_{i+1}$, where \underline{e}_{i+1} is the unit vector with 1 in the $(i+1)$ th place and zero elsewhere, and for the generator of C_i a rotation through $\frac{2\pi}{n_i}$ in the plane of the vectors $\underline{e}_{2i+s}, \underline{e}_{2i+s+1}$.

The Frucht graphs for some abelian groups, are shown affine symmetrically embedded in the plane (but not always in the above manner) in fig. 6.8.1.

6.9 It is by no means obvious whether every finitely generated countable group has a Frucht graph which can be affine symmetrically embedded in the plane. The free product $C_2 * C_2$ of two cyclic groups of order 2 (see (Le 1) p 161) certainly has such an embedding in the plane, as shown in fig. 6.9.1, but it is doubtful whether the free product $C_2 * C_2 * C_2$, part of whose Frucht graph is shown in fig. 6.9.2 can be affine symmetrically embedded in any real space.

6.10 It is worth mentioning that each of the seven groups of strip isometries and the seventeen plane isometry groups can be represented as the group of a graph orthogonal

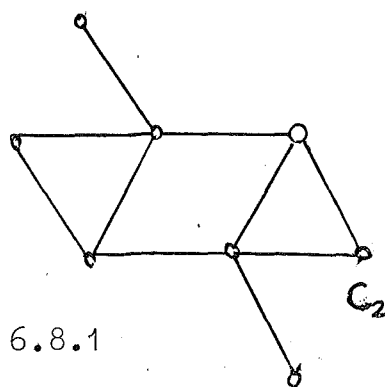
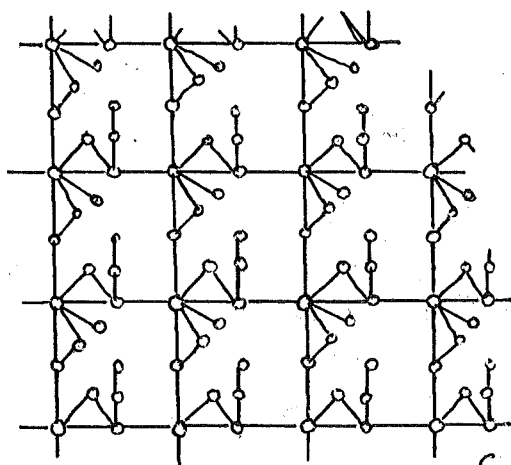
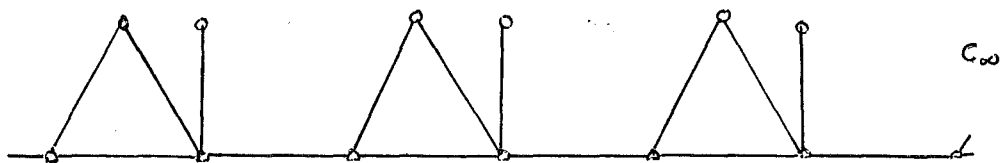


fig. 6.8.1

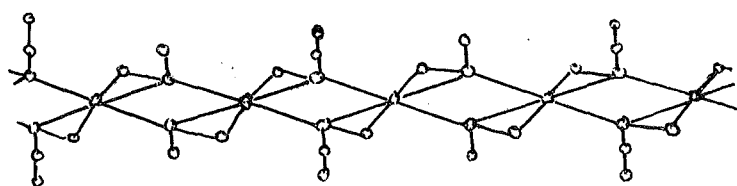
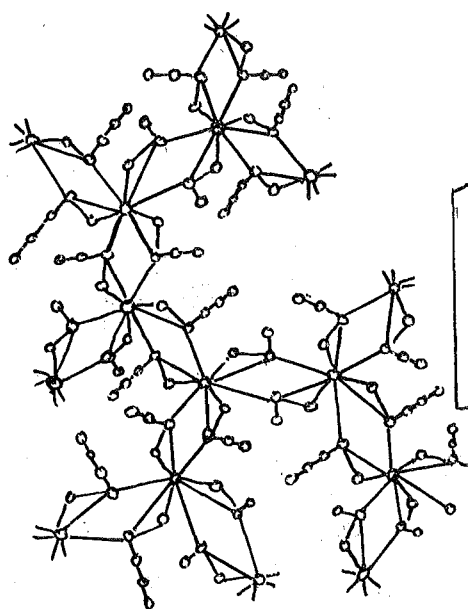
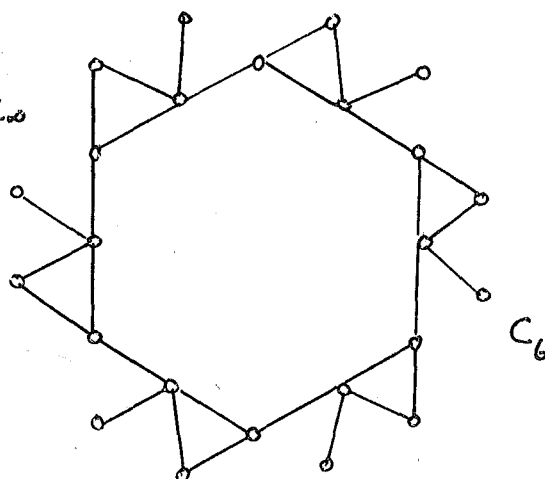


fig. 6.9.1

fig. 6.9.2

affine symmetrically embedded in the plane. This result is stronger than that each group can be represented by a graph with automorphism group isomorphic to it, for several of these groups are isomorphic.

This final remark links the content of these two chapters with the representation of groups by plane figures, which has been a feature of group theory since its early years. The technique of linear and affine symmetric embeddings is indeed just a more flexible embodiment of the same idea. It has the advantage too that we can go a long way towards providing efficient methods of constructing such embeddings for all finite and many of the most interesting infinite groups.

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TERMINOLOGY

The following terms used in this thesis are not in general use. 'D' refers to a formal definition, 'Th' to a theorem in which the term is first used. Other references to section.

automorphism closed D2.6.3
 automorphism closure D2.6.3
 cactus 3.6
 degree of freedom D2.13.2
 dependent transitivity class 5.10
 determining set D2.3.1
 equimomental D2.11.1
 fixed point free 2.6
 fixed vertex 2.6
 fixing subgroup 2.6
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 isosymmetric D3.4.1
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